Chapter 6 INCOMPRESSIBLE INVISCID FLOW

All real fluids possess viscosity. However in many flow cases it is reasonable to neglect the effects of viscosity.

It is useful to investigate the dynamics of an ideal fluid that is incompressible and has zero viscosity.

The analysis of ideal fluid motions is simpler than that for viscous flows because no shear stresses are present in inviscid flow. Normal stresses are the only presses that must be considered in the analysis.

The normal stress in an inviscid flow is the negative of the thermodynamic pressure, \( \sigma_{nn} = -p \).

6.1 Momentum Equation for Frictionless Flow: Euler’s Equation

The equations of motion for frictionless flow, called Euler’s
equations. \((\mu=0, \sigma_{xx}=\sigma_{yy}=\sigma_{zz}=-p\text{ in Navier-Stokes equation})\)

\[
\rho g_x - \frac{\partial p}{\partial x} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \tag{6.1a}
\]

\[
\rho g_y - \frac{\partial p}{\partial y} = \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \tag{6.1b}
\]

\[
\rho g_z - \frac{\partial p}{\partial z} = \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \tag{6.1c}
\]

◎ Vector Form:

\[
\rho \vec{g} - \nabla p = \rho \frac{D\vec{V}}{Dt} \tag{6.3}
\]

◎ In cylindrical coordinates:
6.2 Euler’s Equations in Streamline Coordinates

In describing the motion of a fluid particle in a steady (unsteady) flow, the distance along a streamline is a logical coordinate to use in writing the equations of motion.

For simplicity, consider the flow in the yz plane shown in Fig. 6.1. The equations of motion are to be written in terms of the coordinate \( s \), distance along a streamline, and the coordinate \( n \),
distance normal to the streamline.

Apply Newton’s 2nd law in the streamwise (the s) direction:

\[
\left( p - \frac{\partial p}{\partial s} \frac{ds}{2} \right) dn dx - \left( p + \frac{\partial p}{\partial s} \frac{ds}{2} \right) dn dx - \rho g \sin \beta \, ds \, dn \, dx = \rho a_s \, ds \, dn \, dx
\]

where \( \beta \) is the angle between the tangent to the streamline and the horizontal, and \( a_s \) is the acceleration of the fluid particle along the streamline.
Fig. 6.1  Fluid particle moving along a streamline.
we obtain
\[- \frac{\partial p}{\partial s} - \rho g \sin \beta = \rho a_s\]

Since \(\sin \beta = \frac{\partial z}{\partial s}\), we can write
\[-\frac{1}{\rho} \frac{\partial p}{\partial s} - g \frac{\partial z}{\partial s} = a_s\]

Along any streamline \(V = V(s, t)\), and the total acceleration of a fluid particle in the streamwise direction is given by
\[a_s = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial s}\]

Euler’s equation in the streamwise direction with the \(z\) axis directed vertically upward is then
\[-\frac{1}{\rho} \frac{\partial p}{\partial s} - g \frac{\partial z}{\partial s} = \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial s}\]  

\(\circ\) For steady flow, and neglecting body forces:
Equation 6.5 b indicates that a decrease in velocity is accompanied by an increase in pressure and conversely.

Apply Newton’s 2\textsuperscript{nd} law in the n direction:

\[
\left( p - \frac{\partial p}{\partial n} \frac{dn}{2} \right) ds\, dx - \left( p + \frac{\partial p}{\partial n} \frac{dn}{2} \right) ds\, dx - \rho g \cos \beta \, dn\, dx\, ds = \rho a_n \, dn\, dx\, ds
\]

where \( \beta \) is the angle between the n direction and the vertical, and \( a_n \) is the acceleration of the fluid particle in the n direction.

We obtain
\[-\frac{\partial p}{\partial n} - \rho g \cos \beta = \rho a_n\]

Since \(\cos \beta = \frac{\partial z}{\partial n}\), we write

\[-\frac{1}{\rho} \frac{\partial p}{\partial n} - g \frac{\partial z}{\partial n} = a_n\]

The normal acceleration of the fluid element is toward the center of curvature of the streamline, in the minus \(n\) direction; thus in the coordinate system of Fig. 6.1, the familiar centripetal acceleration is written

\[a_n = \frac{-V^2}{R}\]

for steady flow, where \(R\) is the radius of curvature of the streamline. Then, Euler’s equation normal to the streamline is written for steady flow as

\[\frac{1}{\rho} \frac{\partial p}{\partial n} + g \frac{\partial z}{\partial n} = \frac{V^2}{R}\]  

(6.6a)
For steady flow in a horizontal plane:

\[
\frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{V^2}{R} \tag{6.6b}
\]

Equation 6.6b indicates that pressure increases in the direction outward from the center of curvature of the streamlines. In regions where the streamlines are straight, the radius of curvature, \( R \), is infinite and there is no pressure variation normal to the streamlines.

Ex. 6.1 Flow in a bend: Compute the approximate flow rate.
Apply Euler’s $n$ component equation across flow streamlines.

Basic equation: \[
\frac{\partial p}{\partial r} = \frac{\rho V^2}{r}
\]

Assumptions:
1. Frictionless flow
2. Incompressible flow
3. Uniform flow at measurement section
6.3 Bernoulli Equation – Integration of Euler’s Equation along a streamline for Steady Flow

6.3.1 Derivation Using Streamline Coordinates

Euler’s equation for steady flow along a streamline (from Eq. 6.5a) is

$$ -\frac{1}{\rho} \frac{\partial p}{\partial s} - g \frac{\partial z}{\partial s} = V \frac{\partial V}{\partial s} \quad (6.7) $$

If a fluid particle moves a distance, $ds$, along a streamline, then

$$ \frac{\partial p}{\partial s} ds = dp \quad \text{(the change in pressure along s)} $$

$$ \frac{\partial z}{\partial s} ds = dz \quad \text{(the change in elevation along s)} $$

$$ \frac{\partial V}{\partial s} ds = dV \quad \text{(the change in speed along s)} $$
Thus, after multiplying Eq. 6.7 by $ds$, we can write

\[-\frac{dp}{\rho} - g\,dz = V\,dV\quad \text{or} \quad \frac{dp}{\rho} + V\,dV + g\,dz = 0 \quad \text{(along } s)\]

Integration of this equation gives

\[\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad \text{(along } s)\] \hspace{1cm} (6.8)

Before Eq. 6.8 can be applied, we must specify the relation between pressure and density. For the special case of incompressible flow, $\rho =$ constant, and Eq. 6.8 becomes the Bernoulli equation.

\[
\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \hspace{1cm} (6.9)
\]

Restrictions:

1. Steady flow
2. Incompressible flow
3. Frictionless flow
4. Flow along a streamline
The Bernoulli equation is powerful and useful because it relates pressure changes to velocity and elevation changes along a streamline. However, it gives correct results only when applied to a flow situation whenever all four of the restrictions are reasonable. In general, the Bernoulli constant in Eq. 6.9 has different values along different streamlines. For the case of irrotational flow, the constant has a single value throughout the entire flow field.

6.3.2 Derivation Using Rectangular Coordinates

For steady flow, Euler’s equation in rectangular coordinates can be expressed as

$$\frac{1}{\rho} \nabla p - g \hat{k} = \frac{D\vec{V}}{Dt} = u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} = (\vec{V} \cdot \nabla)\vec{V}$$  (6.10)
If we take the dot product of the terms in Eq. 6.10 with displacement $d\vec{s}$ along the streamline, we obtain a scalar equation relating pressure, speed, and elevation along the streamline. Taking the dot product of $d\vec{s}$ with Eq. 6.10 gives

$$-\frac{1}{\rho} \nabla p \cdot d\vec{s} - g\hat{k} \cdot d\vec{s} = (\vec{V} \cdot \nabla)\vec{V} \cdot d\vec{s}$$  \hspace{1cm} (6.11)

where

$$d\vec{s} = dx\hat{i} + dy\hat{j} + dz\hat{k} \quad \text{(along s)}$$

Now we evaluate each of the three terms in Eq. 6.11

$$-\frac{1}{\rho} \nabla p \cdot d\vec{s} = -\frac{1}{\rho} \left[ \hat{i} \frac{\partial p}{\partial x} + \hat{j} \frac{\partial p}{\partial y} + \hat{k} \frac{\partial p}{\partial z} \right] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= -\frac{1}{\rho} \left[ \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz \right] \quad \text{(along s)}$$

$$-\frac{1}{\rho} \nabla p \cdot d\vec{s} = -\frac{1}{\rho} dp \quad \text{(along s)}$$

and

$$-g\hat{k} \cdot d\vec{s} = -g\hat{k} \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= -g dz \quad \text{(along s)}$$
Using a vector identity, we can write the third term as

\[(\vec{V} \cdot \nabla)\vec{V} \cdot d\vec{s} = \left[ \frac{1}{2} \nabla(\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V}) \right] \cdot d\vec{s} \]

\[= \left\{ \frac{1}{2} \nabla(\vec{V} \cdot \vec{V}) \right\} \cdot d\vec{s} - \left\{ \vec{V} \times (\nabla \times \vec{V}) \right\} \cdot d\vec{s} \]

The last term on the right side of this equation is zero, since \(\vec{V}\) is parallel to \(d\vec{s}\). Consequently,

\[(\vec{V} \cdot \nabla)\vec{V} \cdot d\vec{s} = \frac{1}{2} \nabla(\vec{V} \cdot \vec{V}) \cdot d\vec{s} = \frac{1}{2} \nabla(V^2) \cdot d\vec{s} \hspace{1cm} \text{(along } s)\]

\[= \frac{1}{2} \left[ \hat{i} \frac{\partial V^2}{\partial x} + \hat{j} \frac{\partial V^2}{\partial y} + \hat{k} \frac{\partial V^2}{\partial z} \right] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}] \]

\[= \frac{1}{2} \left[ \frac{\partial V^2}{\partial x} dx + \frac{\partial V^2}{\partial y} dy + \frac{\partial V^2}{\partial z} dz \right] \]

\[(\vec{V} \cdot \nabla)\vec{V} \cdot d\vec{s} = \frac{1}{2} d(V^2) \hspace{1cm} \text{(along } s)\]
Substituting these three terms into Eq. 6.11 yields
\[
\frac{dp}{\rho} + \frac{1}{2} d(V^2) + g \, dz = 0 \quad \text{(along } s)\]

Integrating this equation, we obtain
\[
\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad \text{(along } s)\]

If the density is constant, we obtain the Bernoulli equation
\[
\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}\]

As expected, we see that the last two equations are identical to Eqs. 6.8 and 6.9 derived previously using streamline coordinates.
The Bernoulli equation, derived using rectangular coordinates, is still subject to the restrictions: (1) steady flow, (2) incompressible flow, (3) frictionless flow, and (4) flow along a streamline.

6.3.3 Static, Stagnation, and Dynamic Pressures

The pressure, $p$, which we have used in deriving the Bernoulli equation, E. 6.9, is the thermodynamic pressure; is commonly called the static pressure.

The static pressure is that pressure which would be measured by an instrument moving with the flow. However, such a measurement is rather difficult to make in a practical situation!

How do we measure static pressure experimentally?

In Section 6-2 we showed that there was no pressure variation
normal to straight streamlines. This fact makes it possible to measure the static pressure in a flowing fluid using a wall pressure tap, placed in a region where the flow streamlines are straight, as shown in Fig. 6.2a.

Fig. 6.2 Measurement of static pressure.

© The pressure tap is a small hole, drilled carefully in the wall,
with its axis perpendicular to the surface.

- If the hole is perpendicular to the duct wall and free from burrs, accurate measurements of static pressure can be made by connecting the tap to a suitable pressure-measuring instrument.

- In a fluid stream far from a wall, or where streamlines are curved, accurate static pressure measurements can be made by careful use of a static pressure probe, shown in Fig. 6.2b.

- Such probes must be designed so that the measuring holes are placed correctly with respect to the probe tip and stem to avoid erroneous results. In use, the measuring section must be aligned with the local flow direction.

- Static pressure probes are available commercially in sizes as small as 1.5 mm (1/16 in.) in diameter.

- The stagnation pressure is obtained when a flowing fluid is
decelerated to zero speed by a frictionless process. In incompressible flow, the Bernoulli equation can be used to related changes in speed and pressure along a streamline for such a process.

\[
\frac{p_0}{\rho} + \frac{V_0^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}
\]

or

\[
p_0 = p + \frac{1}{2} \rho V^2
\]  \hspace{1cm} (6.12)

◎ The term \(1/2\rho V^2\) generally is called the dynamic pressure.
Thus, if the stagnation pressure and static pressure could be measured at a point, Eq. 6.13 would give the local flow speed.

Stagnation pressure is measured in the laboratory using a probe with a hole that faces directly upstream as shown in Fig. 6.3. Such a probe is called a stagnation pressure probe, or pitot tube.
If we knew the stagnation pressure and static pressure at a same point, then the flow speed could be computed form Eq. 6.13. Two possible experimental setups are shown in Fig. 6.4.

In Fig. 6.4a, the static pressure corresponding to point A is read from the wall static pressure tap. The stagnation pressure is...
measured directly at A by the **total head tube**, as shown. (The stem of the total head tube is placed downstream from the measurement location to minimize disturbance of the local flow.)

![Diagram](image)

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Fig. 6.4 Simultaneous measurement of stagnation and static pressures.

◎ Two probes often are combined, as in the **pitot-static tube**
shown in Fig. 6.4b. The inner tube is used to measure the stagnation pressure at point B, while the static pressure at C is sensed using the small holes in the outer tube.

In flow fields where the static pressure variation in the streamwise direction is small, the **pitot-static tube** may be used to infer the speed at point B in the flow by assuming $p_B = p_C$ and using Eq. 6.13.

Remember that the **Bernoulli equation** applies only for **incompressible flow** (Mach number, $M \leq 0.3$).

Ex. 6.2 Pitot Tube: Find the flow speed.
6.3.4 Applications

\[ \frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 \]  

(6.14)

where subscripts 1 and 2 represent any two points on a streamline.
Ex. 6.3 Nozzle Flow: Find $p_1 - p_{\text{atm}}$.

Basic equations:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$= 0 (1)$$

$$0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions:

1. Steady flow
2. Incompressible flow
3. Frictionless flow
4. Flow along a streamline
5. $z_1 = z_2$
6. Uniform flow at sections $\text{\textcircled{1}}$ and $\text{\textcircled{2}}$
Ex. 6.4 Flow through a Siphon: Find (a) Speed of water leaving as a free jet, (b) Pressure at point A in the flow.

Ex. 6.5 Flow under a Sluice Gate: Find (a) V2, (b) Q/w (ft³/s per
Ex. 6.6 Bernoulli Equation in Translating Reference Frame: Find $p_{0A}$, $p_B$
$V_{\text{air}} = 0$

$V_w = 150 \text{ km/hr}$

$V_B = 60 \text{ m/s}$ (relative to wing)

$V_{\text{air}} = V_w = 150 \text{ km/hr}$
6.3.5 Cautions on Use of the Bernoulli Equation

◎ Flow through nozzle was modeled well by the Bernoulli equation. Because the pressure gradient in a nozzle is favorable, there is no separation and boundary layers on the walls remain thin. Friction has a negligible effect on the low velocity profile, so 1-D flow is a good model.

◎ A diverging passage or sudden expansion should not be
modeled using the Bernoulli equation. Adverse pressure gradients cause rapid growth of boundary layers, severely distorted velocity profiles, and possible flow separation. 1-D flow is a poor model for such flows.

○ The hydraulic jump is an example of an open-channel flow with adverse pressure gradient. Flow through a hydraulic jump is mixed violently, making it impossible to identify streamlines. Thus the Bernoulli equation cannot be used to model flow through a hydraulic jump.

○ The Bernoulli equation cannot be applied through a machine such as propeller, pump, or windmill. It is impossible to have locally steady flow or to identify streamlines during flow through a machine.

○ Temperature changes can cause significant changes in density
of a gas, even for low-speed flow. Thus the Bernoulli equation could not be applied to air flow through a heating element (e.g., of a hand-held hair dryer) where temperature changes are significant.

6.4 Relation Between The 1st Law of Thermodynamics and the Bernoulli Equation

◎ The Bernoulli equation, Eq. 6.9, was obtained by integrating Euler’s equation along a streamline for steady, incompressible, frictionless flow. Thus Eq. 6.9 was derived from the momentum equation for a fluid particle.

◎ An equation identical in form to Eq. 6.9 (although requiring very different restrictions) may be obtained from the 1st law of thermodynamics.
Consider steady flow in the absence of shear forces. We choose a control volume bounded by streamlines along its periphery. Such a boundary, shown in Fig. 6.5, often is called a stream tube.

Fig. 6.5  Flow through a stream tube.
Basic equation:

\[
\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV + \int_{CS} (e + pv) \rho \vec{V} \cdot d\vec{A} \quad (4.57)
\]

\[
e = u + \frac{V^2}{2} + gz
\]

Restrictions:
1. \( \dot{W}_s = 0 \)
2. \( \dot{W}_{\text{shear}} = 0 \)
3. \( \dot{W}_{\text{other}} = 0 \)
4. Steady flow
5. Uniform flow and properties at each section
Under these restrictions, Eq. 4.57 becomes

\[ 0 = \left( u_1 + p_1 v_1 + \frac{V_1^2}{2} + g z_1 \right) \left\{ -|\rho_1 V_1 A_1| \right\} \]

\[ + \left( u_2 + p_2 v_2 + \frac{V_2^2}{2} + g z_2 \right) \left\{ |\rho_2 V_2 A_2| \right\} - \dot{Q} \]

But from continuity under these restrictions,

\[ = 0(4) \]

\[ 0 = \frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} \]

or

\[ 0 = \left\{ -|\rho_1 V_1 A_1| \right\} + \left\{ |\rho_2 V_2 A_2| \right\} \]

That is,

\[ \dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2 \]
Also
\[ \dot{Q} = \frac{\delta Q}{dt} = \frac{\delta Q}{dm} \frac{dm}{dt} = \frac{\delta Q}{dm} \dot{m} \]

Thus, from the energy equation,
\[ 0 = \left[ \left( p_2v_2 + \frac{V_2^2}{2} + gz_2 \right) - \left( p_1v_1 + \frac{V_1^2}{2} + gz_1 \right) \right] \dot{m} + \left( u_2 - u_1 - \frac{\delta Q}{dm} \right) \dot{m} \]

or
\[ p_1v_1 + \frac{V_1^2}{2} + gz_1 = p_2v_2 + \frac{V_2^2}{2} + gz_2 + \left( u_2 - u_1 - \frac{\delta Q}{dm} \right) \]
Under the additional assumption (6) of incompressible flow, $v_1 = v_2 = 1/\rho$ and hence

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \left( u_2 - u_1 - \frac{\delta Q}{dm} \right) \quad (6.15)$$

Equation 6.15 would reduce to the Bernoulli equation if the term in parentheses were zero. Thus, under the further restriction,

$$\quad (7) \quad (u_2 - u_1 - \delta Q/dm) = 0$$

the energy equation reduces to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

or

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad (6.16)$$

○ Equation 6.16 is identical in form to the Bernoulli equation, Eq. 6.9.

○ The Bernoulli equation was derived from momentum
considerations (Newton’s 2nd law), and is valid for steady, incompressible, frictionless flow along a streamline.

Equation 6.16 was obtained by applying the 1st law of thermodynamics to a stream tube control volume, subject to restrictions 1 through 7 above.

Thus the Bernoulli equation (Eq. 6.9) and the identical form of the energy equation (Eq. 6.16) were developed from entirely different models, coming from entirely different basic concepts, and involving different restrictions.

Note that the restriction 7 \((u_2-u_1-\delta Q/dm=0)\) was necessary to obtain the Bernoulli equation form the 1st law of thermodynamics.

This restriction can be satisfied if \(\delta Q/dm\) is zero (there is no heat transfer to the fluid) and \(u_2=u_1\) (there is no change in the
internal thermal energy of the fluid).

◎ The restriction also is satisfied if \((u_2-u_1)\) and \(\delta Q/dm\) are nonzero provided that the two terms are equal (this is true for incompressible frictionless flow).

◎ For the special case considered in this section it is true that the 1st law of thermodynamics reduces to the Bernoulli equation.

◎ The Bernoulli equation was obtained by integrating the differential form of Newton’s 2nd law (Euler’s equation) for steady, incompressible, frictionless flow along a streamline. Each term in the Bernoulli equation has dimensions of energy per unit mass.

◎ The Bernoulli equation may be viewed as a mechanical energy balance. In those cases wherein there is no conversion of mechanical to thermal energy, mechanical energy and thermal
energy are separately conserved.

For these cases, the 1st law of thermodynamics and Newton’s 2nd law do not yield separate information. However, in general, the 1st law of thermodynamics and Newton’s 2nd law are independent equations that must be satisfied separately.

Ex. 6.7 Internal energy and heat transfer in frictionless incompressible flow: \( u_2 - u_1 = \delta Q / dm \)

Ex. 6.8 frictionless flow with heat transfer:
Basic equations: \[ \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \]
For steady, frictionless, incompressible flow along a streamline, we have shown that the 1\textsuperscript{st} law of thermodynamics reduces to the Bernoulli equation. From Eq. 6.16 we conclude that there is no loss of mechanical energy in such a flow.

Often it is convenient to represent the mechanical energy level
of a flow graphically.

\[
\frac{p}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant} \quad (6.17)
\]

- \( \frac{p}{\rho g} \), the head due to local static pressure
- \( \frac{V^2}{2g} \), the head due to local dynamic pressure
- \( z \), the elevation head
- \( H \), the total head for the flow

◎ The **energy grade line (EGL)** represents the total head height. Liquid would rise to the EGL height in a total head tube placed in the flow.

◎ The **hydraulic grade line (HGL)** height represents the sum of
the elevation and static pressure heads, $z + \frac{p}{\rho g}$. In a static pressure tap attached to the flow conduit, liquid would rise to the HGL height.

The difference in heights between the EGL and the HGL represents the dynamic (velocity) head, $V^2/2g$. 
Fig. 6.6  Energy and hydraulic grade lines for frictionless flow.
①: the free surface in the large reservoir. There the velocity is negligible and the pressure is atmospheric.

① ⇒ ②: The velocity head increases from zero to $\frac{V_2^2}{2g}$ as the liquid accelerates into the first section of constant-diameter tube.

② ⇒ ③: The velocity increase again in the reducer between ② and ③.

③ ⇒ ④: The velocity become constant between ③ and ④.

④: At the free discharge at section ④, the static head is zero.

6.5 Unsteady Bernoulli Equation- Integration of Euler’s Equation Along a Streamline

◎ The momentum equation for frictionless flow was found to be
\[
\frac{1}{\rho} \nabla p - g \hat{k} = \frac{D\vec{V}}{Dt}
\]  \hspace{1cm} (6.3)

It can be converted to a scalar equation by taking the dot product with \(d\vec{s}\), where \(d\vec{s}\) is an element of distance along a streamline. Thus

\[
-\frac{1}{\rho} \nabla p \cdot d\vec{s} - g \hat{k} \cdot d\vec{s} = \frac{D\vec{V}}{Dt} \cdot d\vec{s} = \frac{DV}{Dt} ds = V \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} ds \hspace{1cm} (6.18)
\]

\[\nabla p \cdot d\vec{s} = dp \quad \text{(the change in pressure along s)}\]

\[\hat{k} \cdot d\vec{s} = dz \quad \text{(the change in z along s)}\]

\[\frac{\partial V}{\partial s} ds = dV \quad \text{(the change in V along s)}\]
Substituting into Eq. 6.18, we obtain

\[-\frac{d \rho}{\rho} - g \, dz = V \, dV + \frac{\partial V}{\partial t} \, ds\]  \hspace{1cm} (6.19)

Integrating along a streamline from point 1 to point 2 yields

\[\int_{1}^{2} \frac{d \rho}{\rho} + \frac{V_{2}^{2} - V_{1}^{2}}{2} + g(z_{2} - z_{1}) + \int_{1}^{2} \frac{\partial V}{\partial t} \, ds = 0\]  \hspace{1cm} (6.20)

For incompressible flow, the density is constant. For this special case, Eq. 6.20 becomes

\[\frac{p_{1}}{\rho} + \frac{V_{1}^{2}}{2} + gz_{1} = \frac{p_{2}}{\rho} + \frac{V_{2}^{2}}{2} + gz_{2} + \int_{1}^{2} \frac{\partial V}{\partial t} \, ds\]  \hspace{1cm} (6.21)

Restrictions:

1. Incompressible flow
2. Frictionless flow
3. Flow along a streamline
To evaluate the integral term in Eq. 6.21, the variation in \( \frac{\partial V}{\partial t} \) must be known as a function of \( s \), the distance along the streamline measured from point 1.

Ex. 6.9 Unsteady Bernoulli Equation
$V_2 = 7.67 \tanh (0.639 \, t)$
6.6 Irrotational Flow

An **irrotational flow** is one in which fluid elements moving in the flow field do not undergo any rotation. For \( \vec{\omega} = 0, \nabla \times \vec{V} = 0 \)
In cylindrical coordinates

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (6.22)$$

In cylindrical coordinates

$$\frac{1}{r} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \frac{1}{r} \frac{\partial r V_\theta}{\partial r} - \frac{1}{r} \frac{\partial V_r}{\partial \theta} = 0 \quad (6.23)$$

6.6.1 Bernoulli Equation Applied to Irrotational Flow

If, in addition to being inviscid, steady, and incompressible, the flow field is also irrotational, we can show that Bernoulli’s equation can be applied between any two points in the flow.

To illustrate this, we start with Euler’s equation in vector form,
\[-\frac{1}{\rho} \nabla p - g\hat{k} = (\vec{V} \cdot \nabla)\vec{V} \tag{6.10}\]

Using the vector identity
\[(\vec{V} \cdot \nabla)\vec{V} = \frac{1}{2} \nabla(\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V})\]

we see for irrotational flow, where \(\nabla \times \vec{V} = 0\), that
\[(\vec{V} \cdot \nabla)\vec{V} = \frac{1}{2} \nabla(\vec{V} \cdot \vec{V})\]

and Euler’s equation for irrotational flow can be written as
\[-\frac{1}{\rho} \nabla p - g\hat{k} = \frac{1}{2} \nabla(\vec{V} \cdot \vec{V}) = \frac{1}{2} \nabla(V^2) \tag{6.24}\]
Consider a displacement in the flow field from position $\vec{r}$ to position $\vec{r} + d\vec{r}$; the displacement $d\vec{r}$ is an arbitrary infinitesimal displacement in any direction. Taking the dot product of $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ with each of the terms in Eq. 6.24, we have

$$-\frac{1}{\rho} \nabla p \cdot d\vec{r} - g\hat{k} \cdot d\vec{r} = \frac{1}{2} \nabla(V^2) \cdot d\vec{r}$$

and hence

$$-\frac{dp}{\rho} - gdz = \frac{1}{2}d(V^2)$$

or

$$\frac{dp}{\rho} + \frac{1}{2}d(V^2) + gdz = 0$$

Integrating this equation for incompressible flow gives

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Since $d\vec{r}$ was an arbitrary displacement, Eq. 6.25 is valid between any two points in a steady, incompressible, inviscid flow.
that is also irrotational.

6.6.2 Velocity Potential
◎ In Section 5.2 we formulated the stream function, \( \psi \), which relates the streamlines and mass flow rate in 2-D, incompressible flow.
◎ We can formulate a relation called the potential function, \( \phi \), for a velocity field that is irrotational. To do so, we must use the fundamental vector identity

\[
\text{curl (grad } \phi \text{)} = \nabla \times \nabla \phi = 0
\]

which is valid if \( \phi \) is a scalar function having continuous first and second derivatives.
◎ Then, for an irrotational flow in which \( \nabla \times \vec{V} = 0 \), a scalar
function, $\phi$, must exist such that the gradient of $\phi$ is proportional to the velocity vector, $\vec{V}$.

In order that the positive direction of flow be in the direction of decreasing $\phi$, we define $\phi$ so that

$$\vec{V} = -\nabla \phi$$

(6.27)

Thus

$$u = -\frac{\partial \phi}{\partial x} \quad v = -\frac{\partial \phi}{\partial y} \quad w = -\frac{\partial \phi}{\partial z}$$

(6.28)

With the potential function defined in this way, the irrotationality condition, Eq. 6.22, is satisfied identically.

In cylindrical coordinates
The velocity potential, $\phi$, exist only for irrotational flow. The stream function, $\psi$, satisfies the continuity equation for incompressible flow; the stream function is not subject to the restriction of irrotational flow.

Irrotationality may be a valid assumption for those regions of a flow in which viscous forces are negligible, i.e. a region exists outside the boundary layer in the flow over a solid surface.

The theory for irrotational flow is developed in terms of an imaginary ideal fluid whose viscosity is identically zero. Since,
in an irrotational flow, the velocity field may be defined by the potential function, $\phi$, the theory is often referred to as potential flow theory.

All real fluids possess viscosity, but there are many situations in which the assumption of inviscid flow considerably simplifies the analysis and, at the same time, gives meaningful results.

6.6.3 $\psi$ and $\phi$ for 2-D, Irrotational, Incompressible Flow: Laplace Equation
\[ u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \quad \text{(5.4)} \]

\[ u = -\frac{\partial \Phi}{\partial x} \quad v = -\frac{\partial \Phi}{\partial y} \quad \text{(6.28)} \]

Substituting for \( u \) and \( v \) from \textbf{Eq. 5.4} into the \textbf{irrotationality condition},

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{(6.22)} \]

we obtain

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{(6.30)} \]

Substituting for \( u \) and \( v \) from \textbf{Eq. 6.28} into the \textbf{continuity equation},

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(5.3)} \]

we obtain

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{(6.31)} \]
Equation 6.30 and 6.31 are forms of Laplace’s equation— an equation that arises in many areas of the physical sciences and engineering.

Any function \( \phi \) and \( \psi \) that satisfies Laplace’s equation represent a possible 2-D, incompressible, irrotational flow field.
In Section 5-2 we showed that the stream function, $\psi$, is constant along a streamline. For $\psi = \text{constant}$, $d\psi = 0$ and

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

The slope of a streamline—a line of constant $\psi$—is given by

$$\left(\frac{dy}{dx}\right)_{\psi} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = -\frac{-v}{u} = \frac{v}{u} \quad (6.32)$$

Along a line of constant $\phi$, $d\phi = 0$ and

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

Consequently, the slope of a potential line—a line of constant $\phi$—is given by

$$\left(\frac{dy}{dx}\right)_{\phi} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v} \quad (6.33)$$

Comparing Eqs. 6.32 and 6.33, we see that the slope of a
constant $\psi$ line at any point is the negative reciprocal of the slope of the constant $\phi$ line at that point; lines of constant $\psi$ and constant $\phi$ are orthogonal. This property of potential lines and streamlines is useful in graphical analyses of flow fields.

Ex. 6.10 Velocity potential, $\psi = ax^2 - ay^2$, where $a = 3 \text{ s}^{-1}$. Show that the flow is irrotational. Find: $\phi$

6.6.4 Elementary Plane Flows
◎ A variety of potential flows can be constructed by superposing elementary flow patterns.
◎ The $\phi$ and $\psi$ functions for five elementary 2-D flows- a uniform flow(均勻流), a source(源), a sink(沈), a vortex(渦流),
and a doublet.

◎ **Uniform flow**: inclined at angle $\alpha$ to the x axis, $\psi=(U \cos \alpha)y-(U \sin \alpha)x$, $\phi=-(U \sin \alpha)y-(U \cos \alpha)x$

◎ **Source**: flow is radially outward from the z axis and symmetrical in all directions.

◎ **Sink**: flow is radially inward; a sink is a negative source.

◎ Sources and sinks have no exact physical counterparts. The primary value of the concept of sources and sinks is that, when combined with other elementary flows, they produce flow patterns that adequately represent realistic flows.
Table 6.1  Elementary Plane Flows

Uniform Flow (positive x direction)
\[ u = U \quad \psi = Uy \]
\[ v = 0 \quad \phi = -Ux \]
\[ \Gamma = 0 \] around any closed curve

Source Flow (from origin)
\[ V_r = \frac{q}{2\pi r} \quad \psi = \frac{q}{2\pi} \theta \]
\[ V_\theta = 0 \quad \phi = -\frac{q}{2\pi} \ln r \]

Origin is singular point
\( q \) is volume flow rate per unit depth
\( \Gamma = 0 \) around any closed curve

Sink Flow (toward origin)
\[ V_r = -\frac{q}{2\pi r} \quad \psi = -\frac{q}{2\pi} \theta \]
\[ V_\theta = 0 \quad \phi = \frac{q}{2\pi} \ln r \]

Origin is singular point
\( q \) is volume flow rate per unit depth
\( \Gamma = 0 \) around any closed curve
◎ **Vortex:** the velocity distribution in an irrotational vortex can be determined from Euler’s equation \( \frac{1}{\rho} \frac{dp}{dr} = \frac{V_\theta^2}{r} \) and the Bernoulli equation \( \frac{dp}{\rho} = -V_\theta dV_\theta \).

\[ \Rightarrow \frac{V_\theta^2}{r} dr = -V_\theta dV_\theta \Rightarrow V_\theta dr + rdV_\theta = 0 \Rightarrow rV_\theta = \text{constant} \]

◎ **Doublet:** this flow is produced mathematically by allowing a source and a sink of numerically equal strengths to merge.
### Table 6.1 Elementary Plane Flows (cont’d.)

#### Irrotational Vortex (counter-clockwise, center at origin)

\[
\begin{align*}
V_r &= 0 \\
\psi &= \frac{-K}{2\pi} \ln r \\
V_\theta &= \frac{K}{2\pi r} \\
\phi &= \frac{-K}{2\pi} \theta
\end{align*}
\]

Origin is singular point
\(K\) is strength of the vortex
\(\Gamma = K\) around any closed curve enclosing origin
\(\Gamma = 0\) around any closed curve not enclosing origin

#### Doublet (center at origin)

\[
\begin{align*}
V_r &= -\frac{\Lambda}{r^2} \cos \theta \\
\psi &= -\frac{\Lambda \sin \theta}{r} \\
V_\theta &= -\frac{\Lambda}{r^2} \sin \theta \\
\phi &= -\frac{\Lambda \cos \theta}{r}
\end{align*}
\]

Origin is singular point
\(\Lambda\) is strength of the doublet
\(\Gamma = 0\) around any closed curve
6.6.5 Superposition of Elementary Plane Flows

◎ Both $\phi$ and $\psi$ satisfy Laplace’s equation for flow that is both incompressible and irrotational. Since Laplace’s equation is a linear, homogeneous PDE, solution may be superposed to develop more complex and interesting patterns of flow.

◎ The object of superposition of elementary flow is to produce flow patterns similar to those of practical interest.

◎ Since there is no flow across a streamline, any streamline contour can be imagined to represent a solid surface.

◎ Through the end of the nineteenth century, workers in pure hydrodynamics failed to produce results that agreed with experiment. Potential flows produced body shapes with lift but predicted zero drag (the “d’Alembert paradox”).

◎ Two influences changed this situation: first Prandtl introduced
the BL concept and began to develop the theory, and the second, interest in aeronautics increased dramatically in the early 1900s.

Prandtl showed by mathematical analysis and through elegantly simple experiments that viscous effects are confined to a thin boundary layer on the surface of a body.

Even for real fluids, flow outside the BL behaves as though the fluid had zero viscosity. Pressure gradients from the external flow are impressed on the BL.

The most important input to a calculation of the real fluid flow in the BL is the pressure distribution. Once the velocity field is known from the potential flow solution, the pressure distribution may be calculated.

Two methods of combining elementary flows may be used.

The direct method consists of combining elementary flows.
Distributed line sources, sinks, and “images” may be used to create bodies of arbitrary shape. (Mathematical techniques: complex variables and conformal transformations, can be used to obtain flow fields for interesting geometries)

◎ The inverse methods of superposition calculates the body shape to produce a desired pressure distribution. (A large computer code must be used.)
Table 6.2 Superposition of Elementary Plane Flows

Source and Uniform Flow (flow past a half-body)

\[ \psi = \psi_{so} + \psi_{uf} = \psi_1 + \psi_2 = \frac{q}{2\pi} \theta + Uy = \frac{q}{2\pi} \theta + Ur \sin \theta \]

\[ \phi = \phi_{so} + \phi_{uf} = \phi_1 + \phi_2 = -\frac{q}{2\pi} \ln r - Ux = -\frac{q}{2\pi} \ln r - Ur \cos \theta \]

Source and Sink (equal strength, separation distance on x axis = 2a)

\[ \psi = \psi_{so} + \psi_{si} = \psi_1 + \psi_2 = \frac{q}{2\pi} \theta_1 - \frac{q}{2\pi} \theta_2 = \frac{q}{2\pi} (\theta_1 - \theta_2) \]

\[ \phi = \phi_{so} + \phi_{si} = \phi_1 + \phi_2 = -\frac{q}{2\pi} \ln r_1 + \frac{q}{2\pi} \ln r_2 = \frac{q}{2\pi} \ln \frac{r_2}{r_1} \]

Source, Sink, and Uniform Flow (flow past a Rankine body)

\[ \psi = \psi_{so} + \psi_{si} + \psi_{uf} = \psi_1 + \psi_2 + \psi_3 = \frac{q}{2\pi} \theta_1 - \frac{q}{2\pi} \theta_2 + Uy \]

\[ \psi = \frac{q}{2\pi} (\theta_1 - \theta_2) + Ur \sin \theta \]

\[ \phi = \phi_{so} + \phi_{si} + \phi_{uf} = \phi_1 + \phi_2 + \phi_3 = -\frac{q}{2\pi} \ln r_1 + \frac{q}{2\pi} \ln r_2 - Ux \]

\[ \phi = \frac{q}{2\pi} \ln \frac{r_2}{r_1} - Ur \cos \theta \]
Vortex (clockwise) and Uniform Flow

\[ \psi = \psi_v + \psi_{u_f} = \psi_1 + \psi_2 = \frac{K}{2\pi} \ln r + Uy = \frac{K}{2\pi} \ln r + Ur \sin \theta \]

\[ \phi = \phi_v + \phi_{u_f} = \phi_1 + \phi_2 = \frac{K}{2\pi} \theta - Ux = \frac{K}{2\pi} \theta - Ur \cos \theta \]

Doublet and Uniform Flow (flow past a cylinder)

\[ \psi = \psi_d + \psi_{u_f} = \psi_1 + \psi_2 = -\frac{\Lambda \sin \theta}{r} + Uy = -\frac{\Lambda \sin \theta}{r} + Ur \sin \theta \]

\[ \psi = U \left( r - \frac{\Lambda}{Ur} \right) \sin \theta = Ur \left( 1 - \frac{a^2}{r^2} \right) \sin \theta \quad a = \sqrt{\frac{\Lambda}{U}} \]

\[ \phi = \phi_d + \phi_{u_f} = \phi_1 + \phi_2 = -\frac{\Lambda \cos \theta}{r} - Ux = -\frac{\Lambda \cos \theta}{r} - Ur \cos \theta \]

\[ \phi = -U \left( r + \frac{\Lambda}{Ur} \right) \cos \theta = -Ur \left( 1 + \frac{a^2}{r^2} \right) \cos \theta \]

Doublet, Vortex (clockwise), and Uniform Flow (flow past a cylinder with circulation)

\[ \psi = \psi_d + \psi_v + \psi_{u_f} = \psi_1 + \psi_2 + \psi_3 = -\frac{\Lambda \sin \theta}{r} + \frac{K}{2\pi} \ln r + Uy \]

\[ \psi = -\frac{\Lambda \sin \theta}{r} + \frac{K}{2\pi} \ln r + Ur \sin \theta = Ur \left( 1 - \frac{a^2}{r^2} \right) \sin \theta + \frac{K}{2\pi} \ln r \]

\[ \phi = \phi_d + \phi_v + \phi_{u_f} = \phi_1 + \phi_2 + \phi_3 = -\frac{\Lambda \cos \theta}{r} + \frac{K}{2\pi} \theta - Ux \]

\[ a = \sqrt{\frac{\Lambda}{U}} \quad K < 4\pi a U \quad \phi = -\frac{\Lambda \cos \theta}{r} + \frac{K}{2\pi} \theta - Ur \cos \theta = -Ur \left( 1 + \frac{a^2}{r^2} \right) \cos \theta + \frac{K}{2\pi} \theta \]
Source and Vortex (spiral vortex)

\[
\psi = \psi_{s0} + \psi_{v} = \psi_1 + \psi_2 = \frac{q}{2\pi} \theta - \frac{K}{2\pi} \ln r \\
\phi = \phi_{s0} + \phi_{v} = \phi_1 + \phi_2 = -\frac{q}{2\pi} \ln r - \frac{K}{2\pi} \theta
\]

Sink and Vortex

\[
\psi = \psi_{s1} + \psi_{v} = \psi_1 + \psi_2 = -\frac{q}{2\pi} \theta - \frac{K}{2\pi} \ln r \\
\phi = \phi_{s1} + \phi_{v} = \phi_1 + \phi_2 = \frac{q}{2\pi} \ln r - \frac{K}{2\pi} \theta
\]

Vortex Pair (equal strength, opposite rotation, separation distance on x axis = 2a)

\[
\psi = \psi_{v1} + \psi_{v2} = \psi_1 + \psi_2 = -\frac{K}{2\pi} \ln r_1 + \frac{K}{2\pi} \ln r_2 = \frac{K}{2\pi} \ln \frac{r_2}{r_1} \\
\phi = \phi_{v1} + \phi_{v2} = \phi_1 + \phi_2 = -\frac{K}{2\pi} \theta_1 + \frac{K}{2\pi} \theta_2 = \frac{K}{2\pi} (\theta_2 - \theta_1)
\]
Ex. 6.11 Flow over a cylinder: superposition of doublet and uniform flow.
EXAMPLE PROBLEM 6.11

**GIVEN:** Two-dimensional, incompressible, irrotational flow formed from superposition of a doublet and a uniform flow.

**FIND:**
(a) Stream function and velocity potential.
(b) Velocity field.
(c) Stagnation points.
(d) Cylinder surface.
(e) Surface pressure distribution.
(f) Drag force on the circular cylinder.
(g) Lift force on the circular cylinder.

**SOLUTION:**
Stream functions may be added because the flow field is incompressible and irrotational. Thus from Table 6.1, the stream function for the combination is

\[ \psi = \psi_d + \psi_{uf} = -\frac{\Lambda \sin \theta}{r} + Ur \sin \theta \]

The velocity potential is

\[ \phi = \phi_d + \phi_{uf} = -\frac{\Lambda \cos \theta}{r} - Ur \cos \theta \]

The corresponding velocity components are obtained using Eqs. 6.30 as

\[ V_r = -\frac{\partial \phi}{\partial r} = -\frac{\Lambda \cos \theta}{r^2} + U \cos \theta \]

\[ V_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\Lambda \sin \theta}{r^2} - U \sin \theta \]
The velocity field is
\[
\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta = \left(-\frac{\Lambda \cos \theta}{r^2} + U \cos \theta \right) \hat{e}_r + \left(-\frac{\Lambda \sin \theta}{r^2} - U \sin \theta \right) \hat{e}_\theta
\]

Stagnation points are where \( \vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta = 0 \)

\[
V_r = -\frac{\Lambda \cos \theta}{r^2} + U \cos \theta = \cos \theta \left(U - \frac{\Lambda}{r^2}\right)
\]

Thus \( V_r = 0 \) when \( r = \sqrt{\frac{\Lambda}{U}} = a \). Also,

\[
V_\theta = -\frac{\Lambda \sin \theta}{r^2} - U \sin \theta = -\sin \theta \left(U + \frac{\Lambda}{r^2}\right)
\]

Thus \( V_\theta = 0 \) when \( \theta = 0, \pi \).

Stagnation points are \((r, \theta) = (a, 0), (a, \pi)\).  

Note that \( V_r = 0 \) along \( r = a \), so this represents flow around a circular cylinder, as shown in Table 6.2. Flow is irrotational, so the Bernoulli equation may be applied between any two points. Applying the equation between a point far upstream and a point on the surface of the cylinder (neglecting elevation differences), we obtain

\[
\frac{p_\infty}{\rho} + \frac{U^2}{2} = \frac{p}{\rho} + \frac{V^2}{2}
\]

Thus,

\[
p - p_\infty = \frac{1}{2} \rho(U^2 - V^2)
\]
Along the surface, \( r = a \), and

\[
V^2 = V_\theta^2 = \left( -\frac{\Lambda}{a^2} - U \right)^2 \sin^2 \theta = 4U^2 \sin^2 \theta
\]

since \( \Lambda = Ua^2 \). Substituting yields

\[
p - p_\infty = \frac{1}{2} \rho (U^2 - 4U^2 \sin^2 \theta) = \frac{1}{2} \rho U^2 (1 - 4 \sin^2 \theta)
\]

or

\[
\frac{p - p_\infty}{\frac{1}{2} \rho U^2} = 1 - 4 \sin^2 \theta
\]

Drag is the force component parallel to the freestream flow direction. The drag force is given by

\[
F_D = \int_A -p \, dA \cos \theta = \int_0^{2\pi} -pa \, d\theta \, b \cos \theta
\]

since \( dA = a \, d\theta \, b \), where \( b \) is the length of the cylinder normal to the diagram. Substituting \( p = p_\infty + \frac{1}{2} \rho U^2 (1 - 4 \sin^2 \theta) \),

\[
F_D = \int_0^{2\pi} -p_\infty ab \cos \theta \, d\theta + \int_0^{2\pi} -\frac{1}{2} \rho U^2 (1 - 4 \sin^2 \theta) \, ab \cos \theta \, d\theta
\]

\[= -p_\infty ab \sin \theta \left[ \int_0^{2\pi} -\frac{1}{2} \rho U^2 \sin \theta \right]_0^{2\pi} + \frac{1}{2} \rho U^2 ab \frac{4}{3} \sin^3 \theta \int_0^{2\pi} \]

\[F_D = 0\]
Lift is the force component normal to the freestream flow direction. (By convention, positive lift is an upward force.) The lift force is given by

\[ F_L = \int_A p \, dA (-\sin \theta) = -\int_0^{2\pi} p a \, d\theta \, b \sin \theta \]

Substituting for \( p \) gives

\[ F_L = -\int_0^{2\pi} p_\infty a b \sin \theta \, d\theta - \int_0^{2\pi} \frac{1}{2} \rho U^2 (1 - 4 \sin^2 \theta) a b \sin \theta \, d\theta \]

\[ = p_\infty a b \cos \theta \left[ \int_0^{2\pi} + \frac{1}{2} \rho U^2 a b \cos \theta \int_0^{2\pi} + \frac{1}{2} \rho U^2 a b \left[ \frac{4 \cos^3 \theta}{3} - 4 \cos \theta \right] \right]_0^{2\pi} \]

\[ F_L = 0 \]

This problem illustrates:
- How elementary plane flows can be combined to generate interesting and useful flow patterns.
- d’Alembert’s paradox, that potential flows over a body do not generate drag.

The stream function and pressure distribution are plotted in the Excel workbook.
Ex. 6.12 Flow over a cylinder: superposition of doublet, uniform flow, and clockwise free vortex.

**EXAMPLE PROBLEM 6.12**

**GIVEN:** Two-dimensional, incompressible, irrotational flow formed from superposition of a doublet, a uniform flow, and a clockwise free vortex.

**FIND:**

(a) Stream function and velocity potential.
(b) Velocity field.
(c) Stagnation points.
(d) Cylinder surface.
(e) Surface pressure distribution.
(f) Drag force on the circular cylinder.
(g) Lift force on the circular cylinder.
(h) Lift force in terms of circulation of the free vortex.

**SOLUTION:**

Stream functions may be added because the flow field is incompressible and irrotational. From Table 6.1, the stream function and velocity potential for a clockwise free vortex are
\[ \psi_{fv} = \frac{K}{2\pi} \ln r \quad \phi_{fv} = \frac{K}{2\pi} \theta \]

Using the results of Example Problem 6.11, the stream function for the combination is

\[ \psi = \psi_d + \psi_{uf} + \psi_{fv} \]
\[ \psi = -\frac{\Lambda \sin \theta}{r} + Ur \sin \theta + \frac{K}{2\pi} \ln r \]

The velocity potential for the combination is

\[ \phi = \phi_d + \phi_{uf} + \phi_{fv} \]
\[ \phi = -\frac{\Lambda \cos \theta}{r} - Ur \cos \theta + \frac{K}{2\pi} \theta \]

The corresponding velocity components are obtained using Eqs. 6.30 as

\[ V_r = -\frac{\partial \phi}{\partial r} = -\frac{\Lambda \cos \theta}{r^2} + U \cos \theta \tag{1} \]
\[ V_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\Lambda \sin \theta}{r^2} - U \sin \theta - \frac{K}{2\pi r} \tag{2} \]

The velocity field is

\[ \vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta \]
\[ \vec{V} = \left( -\frac{\Lambda \cos \theta}{r^2} + U \cos \theta \right) \hat{e}_r + \left( -\frac{\Lambda \sin \theta}{r} - U \sin \theta - \frac{K}{2\pi r} \right) \hat{e}_\theta \]
Stagnation points are located where $\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta = 0$. From Eq. 1,

$$V_r = -\frac{\Lambda \cos \theta}{r^2} + U \cos \theta = \cos \theta \left(U - \frac{\Lambda}{r^2}\right)$$

Thus $V_r = 0$ when $r = \sqrt{\frac{\Lambda}{U}} = a$

Cylinder surface

The stagnation points are located on $r = a$. Substituting into Eq. 2 with $r = a$,

$$V_\theta = -\frac{\Lambda \sin \theta}{a^2} - U \sin \theta - \frac{K}{2\pi a}$$

$$= -\frac{\Lambda \sin \theta}{\Lambda/U} - U \sin \theta - \frac{K}{2\pi a}$$

$$V_\theta = -2U \sin \theta - \frac{K}{2\pi a}$$

Thus $V_\theta = 0$ along $r = a$ when

$$\sin \theta = -\frac{K}{4\pi U a} \quad \text{or} \quad \theta = \sin^{-1}\left[-\frac{K}{4\pi U a}\right]$$

Stagnation points: $r = a \quad \theta = \sin^{-1}\left[-\frac{K}{4\pi U a}\right]$
As in Example Problem 6.11, \(V_r = 0\) along \(r = a\), so this flow field once again represents flow around a circular cylinder, as shown in Table 6.2. For \(K = 0\) the solution is identical to that of Example Problem 6.11.

The presence of the free vortex \((K > 0)\) moves the stagnation points below the center of the cylinder. Thus the free vortex alters the vertical symmetry of the flow field. The flow field has two stagnation points for a range of vortex strengths between \(K = 0\) and \(K = 4\pi U a\).

A single stagnation point is located at \(\theta = -\pi/2\) when \(K = 4\pi U a\).

Even with the free vortex present, the flow field is irrotational, so the Bernoulli equation may be applied between any two points. Applying the equation between a point far upstream and a point on the surface of the cylinder we obtain

\[
\frac{p_\infty}{\rho} + \frac{U^2}{2} + gz = \frac{p}{\rho} + \frac{V^2}{2} + gz
\]

Thus, neglecting elevation differences,

\[
p - p_\infty = \frac{1}{2} \rho (U^2 - V^2) = \frac{1}{2} \rho U^2 \left[ 1 - \left( \frac{U}{V} \right)^2 \right]
\]

Along the surface \(r = a\) and \(V_r = 0\), so

\[
V^2 = V_\theta^2 = \left( -2U \sin \theta - \frac{K}{2\pi a} \right)^2
\]

and

\[
\left( \frac{V}{U} \right)^2 = 4 \sin^2 \theta + \frac{2K}{\pi U a} \sin \theta + \frac{K^2}{4\pi^2 U^2 a^2}
\]
Thus

\[ p = p_\infty + \frac{1}{2} \rho U^2 \left( 1 - 4 \sin^2 \theta - \frac{2K}{\pi Ua} \sin \theta - \frac{K^2}{4\pi^2 U^2 a^2} \right) \]

Drag is the force component parallel to the freestream flow direction. As in Example Problem 6.11, the drag force is given by

\[ F_D = \int_A -p \ dA \cos \theta = \int_0^{2\pi} -pa \ d\theta b \cos \theta \]

since \( dA = a \ d\theta \ b \), where \( b \) is the length of the cylinder normal to the diagram.

Comparing pressure distributions, the free vortex contributes only to the terms containing the factor \( K \). The contribution of these terms to the drag force is

\[ \frac{F_{D_{\text{fv}}}}{\frac{1}{2} \rho U^2} = \int_0^{2\pi} \left( \frac{2K}{\pi Ua} \sin \theta + \frac{K^2}{4\pi^2 U^2 a^2} \right) ab \cos \theta \ d\theta \]

\[ = 0 \]

Lift is the force component normal to the freestream flow direction. (Upward force is defined as positive lift.) The lift force is given by

\[ F_L = \int_A -p \ dA \sin \theta = \int_0^{2\pi} -pa \ d\theta b \sin \theta \]
Comparing pressure distributions, the free vortex contributes only to the terms containing the factor $K$. The contribution of these terms to the lift force is

$$\frac{F_{L_{fv}}}{\frac{1}{2} \rho U^2} = \int_0^{2\pi} \left( \frac{2K}{\pi U a} \sin \theta + \frac{K^2}{4\pi^2 U^2 a^2} \right) ab \sin \theta \, d\theta$$

$$= \frac{2K}{\pi U a} \int_0^{2\pi} ab \sin^2 \theta \, d\theta + \frac{K^2}{4\pi^2 U^2 a^2} \int_0^{2\pi} ab \sin \theta \, d\theta$$

$$= \frac{2Kb}{\pi U} \left[ \frac{\theta}{2} - \frac{\sin^2 \theta}{4} \right]_0^{2\pi} - \frac{K^2 b}{4\pi^2 U^2 a} \cos \theta \right]_0^{2\pi}$$

$$\frac{F_{L_{fv}}}{\frac{1}{2} \rho U^2} = \frac{2Kb}{\pi U} \left[ \frac{2\pi}{2} \right] = \frac{2Kb}{U}$$

Thus, $F_{L_{fv}} = \rho U K b$.

The circulation is defined by Eq. 5.18 as

$$\Gamma = \oint \vec{V} \cdot d\vec{s}$$
On the cylinder surface, \( r = a \), and \( \vec{V} = V_\theta \hat{e}_\theta \), so

\[
\Gamma = \int_0^{2\pi} \left( -2U \sin \theta - \frac{K}{2\pi a} \right) \hat{e}_\theta \cdot a \, d\theta \hat{e}_\theta
\]

\[
= - \int_0^{2\pi} 2Ua \sin \theta \, d\theta - \int_0^{2\pi} \frac{K}{2\pi} \, d\theta
\]

\[
\Gamma = -K \text{ (Circulation)}
\]

Substituting into the expression for lift,

\[
F_L = \rho U K b = \rho U (-\Gamma) b = -\rho U \Gamma b
\]

or the lift force per unit length of cylinder is

\[
\frac{F_L}{b} = -\rho U \Gamma \text{ (Unit of lift force)}
\]

This problem illustrates:

- ✔ Once again d’Alembert’s paradox, that potential flows do not generate drag on a body.
- ✔ That the lift per unit length is \(-\rho U \Gamma\). It turns out that this expression for lift is the same for all bodies in an ideal fluid flow, regardless of shape!

The stream function and pressure distribution are plotted in the Excel workbook.