Chapter 4 BASIC EQUATIONS IN INTEGRAL FORM FOR A CONTROL VOLUME

4.1 Basic Laws for a System

The basic laws we will apply are conservation of mass, Newton’s second law, the angular-momentum principle, and the first and second laws of thermodynamics. For converting these system equations to equivalent control volume formulas, it turns out we want to express each of the laws as a rate equation.

**Conservation of Mass:**

\[ M = \text{constant, } \left( \frac{dM}{dt} \right)_\text{system} = 0 \]
where \( M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\mathcal{V}(\text{system})} \rho \, d\mathcal{V} \)

**Newton’s Second Law:**

\[
\vec{F} = \frac{d\vec{P}}{dt}_{\text{system}}, \quad \text{where} \quad \vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V} \, dm = \int_{\mathcal{V}(\text{system})} \vec{V} \, \rho \, d\mathcal{V}
\]

**The Angular-Momentum Principle:**

\[
\vec{T} = \frac{d\vec{H}}{dt}_{\text{system}}, \quad \text{where} \quad \vec{H}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times \vec{V} \, dm = \int_{\mathcal{V}(\text{system})} \vec{r} \times \vec{V} \, \rho \, d\mathcal{V}
\]

Torque can be produced by surface and body forces and also by shafts that cross the system boundary.
\[ \vec{T} = \vec{r} \times \vec{F}_s + \int_{M \text{ (system)}} \vec{r} \times \vec{g} \, dm + \vec{T}_{\text{shaft}} \]

### The First Law of Thermodynamics:

\[ \delta Q - \delta W = dE \Rightarrow \text{In rate form: } \dot{Q} - \dot{W} = \left( \frac{dE}{dt} \right)_{\text{system}} \]

where \( E_{\text{system}} = \int_{M \text{ (system)}} \!edm = \int_{V \text{ (system)}} \!e \rho dV \), \( e = u + \frac{V^2}{2} + gz \).

\( \dot{Q} \) is positive when heat is added to the system from the surroundings; \( \dot{W} \) is positive when works is done by the system on its surroundings.
The Second Law of Thermodynamics:

\[ dS \geq \frac{\delta Q}{T} \Rightarrow \text{In rate form:} \quad \frac{dS}{dt}_{\text{system}} \geq \frac{\dot{Q}}{T} \]

where \( S_{\text{system}} = \int_{M(\text{system})} sdm = \int_{\mathcal{V}(\text{system})} s\rho d\mathcal{V} \)

4.2 Relation of System Derivatives to the Control Volume Formulation

Instead of converting the equations for rates of change of \( M, \bar{P}, \bar{H}, E \) and \( S \) one by one, we let all of them be represented by the symbol \( N \). Hence \( N \) represents the amount of mass, or momentum, or angular momentum, or energy, or entropy of the system.
Corresponding to this extensive property, we will also need the intensive (i.e., per unit mass) property $\eta$. Thus

$$N_{\text{system}} = \int_{M(\text{system})} \eta dm = \int_{\mathcal{V}(\text{system})} \eta \rho d\mathcal{V}$$

- $N = M$, then $\eta = 1$
- $N = \vec{P}$, then $\eta = \vec{V}$
- $N = \vec{H}$, then $\eta = \vec{r} \times \vec{V}$
- $N = E$, then $\eta = e$
- $N = S$, then $\eta = s$

We imagine selecting an arbitrary piece of the flowing fluid at some time $t_0$. This initial shape of the fluid system is chosen as our control volume, which is fixed in space relative to
coordinates \(xyz\). After an infinitesimal time \(\Delta t\) the system will have moved (probably changing shape as it does so) to a new location.

- **Derivation:** The rate of change of \(N_{\text{system}}\) is given by
\[
\frac{dN}{dt}_{\text{system}} \equiv \lim_{\Delta t \to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t}
\]

\[
N_s)_{t_0+\Delta t} = (N_{II} + N_{III})_{t_0+\Delta t} = (N_{CV} - N_I + N_{III})_{t_0+\Delta t}
\]

\[
N_s)_{t_0} = (N_{CV})_{t_0}
\]

\[
\frac{dN}{dt}_{s} = \lim_{\Delta t \to 0} \frac{(N_{CV} - N_I + N_{III})_{t_0+\Delta t} - N_{CV})_{t_0}}{\Delta t}
\]

\[
\frac{dN}{dt}_{s} = \lim_{\Delta t \to 0} \frac{N_{CV})_{t_0+\Delta t} - N_{CV})_{t_0}}{\Delta t} + \lim_{\Delta t \to 0} \frac{N_{III})_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{N_{I})_{t_0+\Delta t}}{\Delta t}
\]

1

2

3
\[
\lim_{\Delta t \to 0} \frac{(N_{CV})_{t_0 + \Delta t} - (N_{CV})_{t_0}}{\Delta t} = \frac{\partial N_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \eta \rho \, d\mathbf{V}
\]

\[
dN_{III}_{t_0 + \Delta t} = (\eta \rho \, d\mathbf{V})_{t_0 + \Delta t}
\]

\[
dN_{III}_{t_0 + \Delta t} = \eta \rho \mathbf{V} \cdot d\mathbf{A} \Delta t
\]
For subregion (1), the velocity vector acts into the control volume, but the area normal always (by convention) points outward, so the scalar product is negative. This concept of the sign of the scalar product is illustrated in the following figures for (a) the general case of an inlet or exit, (b) an exit velocity parallel to the surface normal, and (c) an inlet velocity parallel to the surface normal.

\[ \lim_{\Delta t \to 0} \frac{N_{\text{III}}(t_0 + \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\int_{CS_{\text{III}}} dN_{\text{III}}(t_0 + \Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\int_{CS_{\text{III}}} \eta \rho \vec{V} \cdot d\vec{A} \, \Delta t}{\Delta t} = \int_{CS_{\text{III}}} \eta \rho \vec{V} \cdot d\vec{A} \]

\[ \lim_{\Delta t \to 0} \frac{N_{\text{I}}(t_0 + \Delta t)}{\Delta t} = -\int_{CS_{\text{I}}} \eta \rho \vec{V} \cdot d\vec{A} \]
The two last integrals can be combined because CS\textsubscript{I} and CS\textsubscript{III} constitute the entire control surface.

\[
\frac{dN}{dt}\text{ \_\_\_system} = \frac{\partial}{\partial t} \int_{CV} \eta \rho \, dV + \int_{CS\textsubscript{I}} \eta \rho \vec{V} \cdot d\vec{A} + \int_{CS\textsubscript{III}} \eta \rho \vec{V} \cdot d\vec{A}
\]

The two last integrals can be combined because CS\textsubscript{I} and CS\textsubscript{III} constitute the entire control surface.

\[
\frac{dN}{dt}\text{ \_\_\_system} = \frac{\partial}{\partial t} \int_{CV} \eta \rho \, dV + \int_{CS} \eta \rho \vec{V} \cdot d\vec{A}
\]

It is the fundamental relation between the rate of change of any
arbitrary extensive property, $N$, of a system and the variations of this property associated with a control volume. Some authors refer to above equation as the Reynolds Transport Theorem.

**Physical Interpretation:**

$$\left( \frac{dN}{dt} \right)_{\text{system}}$$ is the rate of change of the system extensive property $N$.

$$\frac{\partial}{\partial t} \int_{\text{CV}} \eta \rho dV$$ is the rate of change of the amount of property $N$ in the control volume.

$$\int_{\text{CS}} \eta \rho \vec{V} \cdot d\vec{A}$$ is the rate at which property $N$ is exiting the control surface.
surface of the control volume. The term $\rho \vec{V} \cdot d\vec{A}$ computes the rate of mass transfer leaving across control surface area element $d\vec{A}$.

Because $\vec{A}$ is always directed outwards, the dot product will be positive when $\vec{V}$ is outward and negative when $\vec{V}$ is inward.

$\vec{V}$ is measured with respect to the control volume: When the control volume coordinates $xyz$ are stationary or moving with a constant linear velocity, the control volume will constitute an inertial frame and the physical laws (specifically Newton’s second law) we have described will apply.

For an accelerating control volume (one whose coordinates $xyz$ are accelerating with respect to an “absolute” set of
coordinates $XYZ), we must modify the form of Newton’s second law.

4.3 Conservation of Mass

The first physical principle to which we apply this conversion from a system to a control volume description is the mass conservation principle: The mass of the system remains constant.

$$\frac{dM}{dt} = 0, \quad N = M, \quad \eta = 1.$$  

$$\Rightarrow \frac{dM}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0.$$
\[ \Rightarrow \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \]

- The first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass flux out through the control surface.
- It indicates the rate of change of mass in the control volume plus the net outflow is zero.
- The mass conservation equation is also called the continuity equation.
- \( \vec{V} \cdot d\vec{A} = V dA \cos \alpha \): It could be positive (outflow, \( \alpha < \pi / 2 \)), negative (inflow, \( \alpha > \pi / 2 \)), or even zero (\( \alpha = \pi / 2 \)).
- **Special Cases:** an incompressible fluid (\( \rho = \text{constant} \))
\[
\Rightarrow \frac{\partial \mathbf{V}}{\partial t} + \int_{\text{CS}} \mathbf{V} \cdot d\mathbf{A} = 0
\]

For incompressible fluids, nondeformable control volume of fixed size and uniform velocity at each inlet and exit.

\[
\sum_{\text{CS}} \mathbf{V} \cdot \mathbf{A} = 0
\]

Above equations are statements of conservation of mass for flow of an incompressible fluid that may be steady or unsteady.

The dimensions of the integrand in above equations are \(L^3/t\). The integral of \(\mathbf{V} \cdot d\mathbf{A}\) over a section of the control surface is commonly called the volume flow rate or volume rate of flow.

The volume flow rate \(Q\), through a section of a control surface
of area $A$, is given by

$$ Q = \int_A \vec{V} \cdot d\vec{A} $$

- The **average velocity** magnitude, $\bar{V}$, at a section is defined as

$$ \bar{V} = \frac{Q}{A} = \frac{1}{A} \int_A \vec{V} \cdot d\vec{A} $$

- For steady, compressible flow through a fixed control volume.

$$ \int_{cs} \rho \vec{V} \cdot d\vec{A} = 0 $$

- When we have uniform velocity at each inlet and exit (steady, compressible flow)

$$ \sum_{cs} \rho \vec{V} \cdot \vec{A} = 0 $$
Ex. 4.1 Mass Flow at a Pipe Junction: Consider the steady flow in a water pipe joint shown in the diagram. The areas are: \( A_1 = 0.2 \text{ m}^2 \), \( A_2 = 0.2 \text{ m}^2 \), and \( A_3 = 0.15 \text{ m}^2 \). In addition, fluid is lost out of a hole at 4, estimated at a rate of 0.1 \( \text{m}^3/\text{s} \). The average speeds at sections 1 and 3 are \( V_1 = 5 \text{ m/s} \) and \( V_3 = 12 \text{ m/s} \), respectively. Find the velocity at section 2.

**Given:** Steady flow of water through the device. \( A_1 = 0.2 \text{ m}^2 \), \( A_2 = 0.2 \text{ m}^2 \), \( A_3 = 0.15 \text{ m}^2 \), \( V_1 = 5 \text{ m/s} \), \( V_3 = 12 \text{ m/s} \), \( \rho = 999 \text{ kg/m}^3 \), Volume flow rate of a hole at 4 = 0.1 \( \text{m}^3/\text{s} \).

**Find:** Velocity at section 2.
Ex. 4.2 Mass Flow Rate in Boundary Layer: The fluid in direct contact with a stationary solid boundary has zero velocity; there is no slip at the boundary. Thus the flow over a flat plate adheres to the plate surface and forms a boundary layer, as depicted below. The flow ahead of the plate is uniform with velocity $V = Ui$; $U=$
30 m/s. The velocity distribution within the boundary layer (0 ≤ y ≤ δ) along cd is approximated as \( u/U = 2(y/\delta) - (y/\delta)^2 \). The boundary-layer thickness at location d is \( \delta = 5 \) mm. The fluid is air with density \( \rho = 1.24 \text{ kg/m}^3 \). Assuming the plate width perpendicular to the paper to be \( w = 0.6 \) m, calculate the mass flow rate across surface \( bc \) of control volume \( abcd \).

**Given:** Steady, incompressible flow over a flat plate, \( \rho = 1.24 \text{ kg/m}^3 \). Width of plate, \( w = 0.6 \) m. Velocity ahead of plate is uniform: \( \vec{V} = U\hat{i} \), \( U = 30 \) m/s. At \( x = x_d \): \( \delta = 5 \) mm, \( u/U = 2(y/\delta) - (y/\delta)^2 \).

**Find:** Mass flow rate across surface \( bc \).
Ex. 4.3 Density Change in Venting Tank: A tank of 0.05 m$^3$ volume contains air at 800 kPa (absolute) and 15°C. At $t=0$, air begins escaping from the tank through a valve with a flow area of 65 mm$^2$. The air passing through the valve has a speed of 300 m/s and a density of 6 kg/m$^3$. Determine the instantaneous rate of change of density in the tank at $t=0$.

Given: Tank of volume $V = 0.05$ m$^3$ contains air at $p=800$ kPa (absolute), $T=15$°C. At $t=0$, air escapes through a valve. Air leaves with speed $V=300$ m/s and density $\rho=6$ kg/m$^3$ through area $A=65$ mm$^2$.

Find: Rate of change of air density in the tank at $t=0$. 
4.4 Momentum Equation for Inertial Control Volume

- Newton’s second law for a system moving relative to an inertial coordinate system

\[ \vec{F} = \frac{d\vec{P}}{dt} \quad \text{system} \]

\[ N = \vec{P}, \quad \eta = \vec{V}, \quad \vec{F} = \vec{F}_s + \vec{F}_b \]

\[ \Rightarrow \frac{d\vec{P}}{dt} \quad \text{system} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho dV + \int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} = \vec{F} \quad \text{on system} \quad = \vec{F} \quad \text{on control volume} \]
\[ \vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t} \int CV \rho \vec{V} dV + \int CS \rho \vec{V} \cdot dA \]

It states that the total force (due to surface and body forces) acting on the control volume leads to a rate of change of momentum within the control volume (the volume integral) and/or a net rate at which momentum is leaving the control volume through the control surface.

The first step will always be to carefully choose a control volume and its control surface so that we can evaluate the volume integral and the surface integral (or summation); each inlet and exit should be carefully labeled, as should the external forces acting.

In fluid mechanics the body force is usually gravity, so
\[
\vec{F}_B = \int_{CV} \rho \vec{g} d\mathbf{V} = \vec{W}_{CV} = M\vec{g}
\]

- In many applications the surface force is due to pressure.
\[
\vec{F}_S = \int_A -p dA
\]

Note that the minus sign is to ensure that we always compute pressure forces acting onto the control surface.

- The momentum equation is a vector equation. We usually write the three scalar components, as measured in the \(xyz\) coordinates of the control volume,
\[
F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t} \int_{CV} u \rho d\mathbf{V} + \int_{CS} u \rho \vec{V} \cdot d\vec{A}
\]
\begin{align*}
F_y &= F_{S_y} + F_{B_y} = \frac{\partial}{\partial t} \int_{CV} v \rho d\vec{V} + \int_{CS} v \rho \vec{V} \cdot d\vec{A} \\
F_z &= F_{S_z} + F_{B_z} = \frac{\partial}{\partial t} \int_{CV} w \rho d\vec{V} + \int_{CS} w \rho \vec{V} \cdot d\vec{A}
\end{align*}

For steady flow, the first term on the right in above equations is zero.

Ex. 4.4 Choice of Control Volume for Momentum Analysis: Water from a stationary nozzle strikes a flat plate as shown. The water leaves the nozzle at 15 m/s; the nozzle area is 0.01 m². Assuming the water is directed normal to the plate, and flows along the plate, determine the horizontal force you need to resist to hold it in place.
Given: Water from a stationary nozzle is directed normal to the plate; subsequent flow is parallel to plate.
Jet velocity; \( \vec{V} = 15\hat{i} \) m/s, Nozzle area; \( A_n = 0.01\text{m}^2 \)
Find: Horizontal force on your hand.
Ex. 4.5 Tank on Scale- Body Force: A metal container 0.61 m high, with an inside cross-sectional area of 0.09 m², weighs 22.2 N when empty. The container is placed on a scale and water flows in through an opening in the top and out through the two equal-area openings in the sides, as shown in the diagram. Under steady flow conditions, the height of the water in the tank is 0.58 m.

\[ A_1 = 0.009 \text{ m}^2, \quad \vec{V}_1 = -3 \hat{j} \text{ m/s}, \quad A_2 = A_3 = 0.009 \text{ m}^2. \]

Your boss claims that the scale will read the weight of the volume
of water in the tank plus the tank weight, i.e., that we can treat this as a simple statics problem. You disagree, claiming that a fluid flow analysis is required. Who is right, and what does the scale indicate?

**Given:** Metal container, of height 0.61 m and cross-sectional area \( A = 0.09 \text{ m}^2 \), weighs 22.2 N when empty. Container rests on scale. Under steady flow conditions water depth is \( h = 0.58 \text{ m} \). Water enters vertically at section 1 and leaves horizontally through sections 2 and 3.

\[ A_1 = 0.009 \text{ m}^2, \quad \vec{V}_1 = -3 \hat{j} \text{ m/s}, \quad A_2 = A_3 = 0.009 \text{ m}^2. \]

**Find:** Scale reading.
Ex. 4.6 Flow through Elbow- Use of Gage Pressures: Water flows steadily through the 90° reducing elbow shown in the diagram. At the inlet to the elbow, the absolute pressure is 220 kPa and the cross-sectional area is 0.01 m². At the outlet, the cross-sectional area is 0.0025 m² and the velocity is 16 m/s. The elbow discharges to the atmosphere. Determine the force required to hold the elbow in place.

Given: Steady flow of water through 90° reducing elbow.
\( p_1 = 220 \text{ kPa (abs)}, \) \( A_1 = 0.01 \text{m}^2, \) \( V_2 = -16 \vec{j} \text{ m/s}, \) \( A_2 = 0.0025 \text{ m}^2 \)

**Find:** Force required to hold elbow in place.

**Ex. 4.7 Flow under a Sluice Gate- Hydrostatic Pressure Force:**
Water in an open channel is held in by a sluice gate. Compare the
horizontal force of the water on the gate (a) when the gate is closed and (b) when it is open (assuming steady flow, as shown). Assume the flow at sections 1 and 2 is incompressible and uniform, and that (because the streamlines are straight there) the pressure distributions are hydrostatic.

**Given:** Flow under sluice gate. Width=$w$.

**Find:** Horizontal force (per unit width) on the closed and open gate.
Ex. 4.8 Conveyor Belt Filling- Rate of Change of Momentum in Control Volume: A horizontal conveyor belt moving at 0.9 m/s receives sand from a hopper. The sand falls vertically from the hopper to the belt at a speed of 1.5 m/s and a flow rate of 225 kg/s (the density of sand is approximately 1580 kg/m³). The conveyor belt is initially empty but begins to fill with sand. If friction in the drive system and rollers is negligible, find the tension required to pull the belt while the conveyor is filling.
Given: Conveyor and hopper shown in sketch.
Find: $T_{belt}$ at the instant shown.

Control Volume moving with Constant Velocity

In the preceding problems, which illustrate applications of the momentum equation to inertial control volumes, we have considered only stationary control volumes. Suppose we have a control volume moving at constant speed. We can set up two coordinate systems: $XYZ$, “absolute,” or stationary (and
therefore inertial), coordinates, and the $xyz$ coordinates attached to the control volume (also inertial because the control volume is not accelerating with respect to $XYZ$).

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \frac{\partial}{\partial t} \int_{CV} \mathbf{V}_{xyz} \rho d\mathbf{V} + \int_{CS} \mathbf{V}_{xyz} \rho \mathbf{V}_{xyz} \cdot d\mathbf{A}$$

Above equation is the formulation of Newton’s second law applied to any inertial control volume (stationary or moving with a constant velocity).

**Ex. 4.10 Vane Moving with Constant Velocity:** The sketch shows a vane with a turning angle of $60^\circ$. The vane moves at constant speed, $U=10\text{m/s}$, and receives a jet of water that leaves a stationary nozzle with speed $V=30\text{m/s}$. The nozzle has an exit area of $0.003$
m². Determine the force components that act on the vane.

**Given:** Vane, with turning angle $\theta=60^\circ$, moves with constant velocity, $\vec{U}=10\hat{i}$ m/s. Water from a constant area nozzle, $A=0.003m^2$, with velocity $\vec{V}=30\hat{i}$ m/s, flows over the vane as shown.  

**Find:** Force components acting on the vane.

![Diagram of vane and water flow](image1)

4.5 Momentum Equation for Control Volume with Rectilinear Acceleration

- Not all control volumes are inertial; for example, a rocket must
accelerate if it is to get off the ground.

It is logical to ask whether Eq. 4.26 (for an inertial CV) can be used for an accelerating control volume.

\[
\vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho \, d\mathcal{V} + \int_{CS} \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\mathcal{A} \quad (4.26)
\]

In relating the system derivatives to the control volume formulation (Eq. 4.25), the flow field, \( \vec{V}(x, y, z, t) \), was specified relative to the control volume’s coordinates \( x, y, \) and \( z \). No restriction was placed on the motion of the \( xyz \) reference frame.

\[
\left( \frac{dN}{dt} \right)_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \eta \rho \, d\mathcal{V} + \int_{CS} \eta \rho \vec{V}_{xyz} \cdot d\mathcal{A} \quad (4.25)
\]
Consequently, Eq. 4.25 is valid at any instant for any arbitrary motion of the coordinates x, y, and z provided that all velocities in the equation are measured relative to the control volume.

The system equation, $\vec{F} = d\vec{P} / dt$ \text{(system)} , is valid only for velocities measured relative to an inertial reference frame.

Eq. 4.26 is not valid for an accelerating control volume.

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\vec{P}_{XYZ}$ of the system to $\vec{P}_{xyz}$ of the system. (Inertial reference frame designated XYZ)

$$\vec{F} = \left( \frac{d\vec{P}_{XYZ}}{dt} \right)_{\text{system}} = \frac{d}{dt} \int_{M(\text{system})} \vec{V}_{XYZ} dm = \int_{M(\text{system})} \frac{d\vec{V}_{XYZ}}{dt} dm$$
The velocities with respect to the inertial (XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation

\[ \vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{V}_{rf} \]

\[ \frac{d\vec{V}_{XYZ}}{dt} = \vec{a}_{XYZ} = \frac{d\vec{V}_{xyz}}{dt} + \frac{d\vec{V}_{rf}}{dt} = \vec{a}_{xyz} + \vec{a}_{rf} \]

\( \vec{a}_{XYZ} \) is the rectilinear acceleration of the system relative to inertial reference frame XYZ,

\( \vec{a}_{xyz} \) is the rectilinear acceleration of the system relative to noninertial reference frame xyz (i.e., relative to the control volume), and

\( \vec{a}_{rf} \) is the rectilinear acceleration of noninertial reference frame xyz (i.e., of the control volume) relative to inertial frame XYZ.

\[ \vec{F} = \int_{M_{system}} \vec{a}_{rf} \, dm + \int_{M_{system}} \frac{d\vec{V}_{xyz}}{dt} \, dm = \frac{d\vec{P}_{xyz}}{dt} \]
Ex. 4.11 Vane Moving with Rectilinear Acceleration: A vane, with turning angle $\theta=60^\circ$, is attached to a cart. The cart and vane, of mass $M=75$ kg, roll on a level track. Friction and air resistance
may be neglected. The vane receives a jet of water, which leaves a stationary nozzle horizontally at $V=35$ m/s. The nozzle exit area is $A=0.003\,\text{m}^2$. Determine the velocity of the cart as a function of time and plot the results.

**Given:** Vane and cart as sketched, with $M=75$ kg.

**Find:** $U(t)$ and plot results.

**Solution:**
G.E. \[ \hat{F}_S + \hat{F}_B - \int_{CV} \vec{a}_{rf} \rho d\mathcal{V} = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho d\mathcal{V} + \int_{CS} \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} \]

Assumptions: (1) \( F_{sx} = 0 \), since no resistance is present; (2) \( F_{bx} = 0 \); (3) Neglect the mass of water in contact with the vane compared to the cart mass; (4) Neglect rate of change of momentum of liquid inside the CV. \( \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho d\mathcal{V} \approx 0 \); (5) Uniform flow at sections 1 and 2. (6) Speed of water stream is not slowed by friction on the vane; (7) \( A_2 = A_1 = A \).

\[ \frac{F_{sx}}{F_{sx}} + \frac{F_{bx}}{F_{bx}} - \int_{CV} a_{rfx} \rho d\mathcal{V} = \frac{\partial}{\partial t} \int_{CV} u_{xyz} \rho d\mathcal{V} + \sum_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot \vec{A} \]
\[- \int_{CV} a_{rf_x} \rho \, dV = \sum_{CS} u_{xyz} \rho \vec{V}_{xyz} \cdot \vec{A} = u_2 (\rho V_2 A_2) + u_1 (-\rho V_1 A_1)\]

\[
\begin{bmatrix}
u_2 = (V - U) \cos \theta, \ u_1 = (V - U), \ V_1 = V_2 = (V - U) \end{bmatrix}
\]

\[= (V - U) \cos \theta \rho (V - U) A - (V - U) \rho (V - U) A\]

\[= \rho (V - U)^2 A (\cos \theta - 1)\]

\[\int_{CV} a_{rf_x} \rho dV = a_{rf_x} M_{CV} = \rho (V - U)^2 A (1 - \cos \theta)\]

\[
\frac{dU}{dt} M_{CV} = \rho (V - U)^2 A (1 - \cos \theta)\]

\[\int_{0}^{U} \frac{dU}{(V - U)^2} = \int_{0}^{t} \frac{\rho A (1 - \cos \theta)}{M_{CV}} dt\]
\[ \frac{U}{V} = \frac{V(1-\cos \theta) \rho A}{M} t = \frac{35(1-\cos 60^\circ)(999)(0.003)}{75} t \]

\[ = \frac{0.699t}{1+0.699t} \]
Ex. 4.12 Rocket Directed Vertically: A small rocket, with an initial mass of 400 kg, is to be launched vertically. Upon ignition the rocket consumes fuel at the rate of 5 kg/s and ejects gas at atmospheric pressure with a speed of 3500 m/s relative to the rocket. Determine the initial acceleration of the rocket and the rocket speed after 10 s, if air resistance is neglected. Comparing this momentum equation for a control volume with rectilinear acceleration to that for a nonaccelerating control volume, Eq. 4.26, we see that the only difference is the presence of one additional term in Eq. 4.33.

**Given:** Small rocket accelerates vertically from rest. Initial mass, $M_0= 400$ kg. Air resistance may be neglected. Rate of fuel consumption, $m_e= 5$ kg/s. Exhaust velocity, $V_e= 3500$ m/s, relative
to rocket, leaving at atmospheric pressure.

Find: (a) Initial acceleration of the rocket; (b) Rocket velocity after 10 s.

Solution:

\[
\text{G.E. } \vec{F}_S + \vec{F}_B - \int_{CV} \vec{a}_{rf} \rho d\mathcal{V} = \frac{\partial}{\partial t} \int_{CV} \vec{V}_{xyz} \rho d\mathcal{V} + \int_{CS} \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\mathcal{A}
\]

Assumptions: (1) Atmospheric pressure acts on all surfaces of the
CV; since air resistance is neglected, $F_{s_y} = 0$; (2) Gravity is the only body force; $g$ is constant; (3) Flow leaving the rocket is uniform, and $V_e$ is constant.

\[
\frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} = 0
\]

\[
F_{s_y} + F_{B_y} - \int_{CV} a_{rf_y} \rho dV = \frac{\partial}{\partial t} \int_{CV} v_{xyz} \rho dV + \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A}
\]

**A**: $F_{B_y} = -gM_{CV}$

\[
\frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V}_{xyz} \cdot d\vec{A} = 0 \Rightarrow \frac{\partial M_{CV}}{\partial t} = -\sum_{CS} \rho \vec{V}_{xyz} \cdot \vec{A} = -\dot{m}_e
\]
\[\Rightarrow \frac{dM_{CV}}{dt} = -\dot{m}_e \quad \text{(Since the mass of the CV is only a function of time.)}\]

\[\int_{M_0}^{M_{CV}} dM_{CV} = -\int_0^t \dot{m}_e dt \Rightarrow M_{CV} - M_0 = -\dot{m}_e t \Rightarrow M_{CV} = M_0 - \dot{m}_e t\]

A: \[F_{By} = -gM_{CV} = -g(M_0 - \dot{m}_e t)\]

B: \[\int \int_{CV} a_{rf_y} \rho dV = -a_{rf_y} M_{CV} = -a_{rf_y} (M_0 - \dot{m}_e t)\]

C: \[\frac{\partial}{\partial t} \int_{CV} v_{xyz} \rho dV \approx 0\]

(1) The unburned fuel and the rocket structure have zero momentum relative to the rocket; (2) The velocity of the gas at the nozzle exit remains constant with time as does the velocity at
various points in the nozzle.

\[ D: \int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} = \sum_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot \vec{A} = -V_e \sum_{CS} \rho \vec{V}_{xyz} \cdot \vec{A} = -V_e \dot{m}_e \]

\[-g \left( M_0 - \dot{m}_e t \right) - a_{rf_y} \left( M_0 - \dot{m}_e t \right) = -V_e \dot{m}_e \]

\[ a_{rf_y} = \frac{V_e \dot{m}_e}{M_0 - \dot{m}_e t} - g \]

\[ a_{rf_y} \bigg|_{t=0} = \frac{V_e \dot{m}_e}{M_0} - g = \frac{(3500)(5)}{400} - 9.81 = 33.9 \text{ m/s}^2 \]

\[ a_{rf_y} = \frac{dV_{CV}}{dt} = \frac{V_e \dot{m}_e}{M_0 - \dot{m}_e t} - g \Rightarrow \int_{V_{CV}}^{0} V_{CV} = \int_{0}^{t} \frac{V_e \dot{m}_e}{M_0 - \dot{m}_e t} dt - \int_{0}^{t} g dt \]
\[ V_{CV} = -V_e \ln \left( \frac{M_0 - m_e t}{M_0} \right) - gt = (-3500) \ln \left[ \frac{400 - (5)(10)}{400} \right] = 369 \text{ m/s} \]

4.6 Momentum Equation for Control Volume with Arbitrary Acceleration (on the Web)

4.7 The Angular-Momentum Principle

There are two obvious approaches we can use to express the angular-momentum principle: We can use an inertial (fixed) XYZ control volume; we can also use a rotating xyz control volume. For each approach we will: start with the principle in its system form (Eq. 4.3a), then write the system angular momentum in terms of XYZ or xyz coordinates, and finally use
Eq. 4.10 to convert from a system to a control volume formulation.

\[ \dot{T} = \left( \frac{dH}{dt} \right)_{\text{system}} \]  

(4.3a)

\[ \frac{dN}{dt} \bigg|_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \eta \rho \, d\mathbf{v} + \int_{CS} \eta \rho \mathbf{V} \cdot d\mathbf{A} \]  

(4.10)

To verify that these two approaches are equivalent, we will use each approach to solve the same problem, in Examples 4.14 and 4.15 (on the Web), respectively.

**Equation for Fixed Control Volume:**

The angular-momentum principle for a system in an inertial frame is
Equation 4.46 is a general formulation of the angular-momentum principle for an inertial control volume. The left side of the equation is an expression for all the torques that act on the control volume. Terms on the right express the
rate of change of angular momentum within the control volume and the net rate of flux of angular momentum from the control volume.

➢ All velocities in Eq. 4.46 are measured relative to the fixed control volume.

Ex. 4.14 Lawn Sprinkler- Analysis using Fixed Control Volume: A small lawn sprinkler is shown in the sketch at right. At an inlet gage pressure of 20 kPa, the total volume flow rate of water through the sprinkler is 7.5 liters per minute and it rotates at 30 rpm. The diameter of each jet is 4 mm. Calculate the jet speed relative to each sprinkler nozzle. Evaluate the friction torque at the sprinkler pivot.

Given: Small lawn sprinkler as shown.
Find: (a) Jet speed relative to each nozzle; (b) Friction torque at pivot.

Solution:

G.E. \[ \frac{\partial}{\partial t} \int_{CV} \rho d\vec{V} + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \]

\[ \vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g} \rho d\vec{V} + T_{shaft} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho d\vec{V} + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \]

Assumptions: (1) Incompressible flow; (2) Uniform flow at each section; (3) \( \omega = \) constant.
\[ \frac{\partial}{\partial t} \int_{CV} \rho dV + \sum_{CS} \rho \vec{V} \cdot \vec{A} = 0 \Rightarrow Q = 2A_{\text{jet}} V_{\text{rel}} \]

\[ \Rightarrow V_{\text{rel}} = \frac{Q}{2A_{\text{jet}}} = \frac{7.5 / (1000)(60)}{2 \pi (0.004)^2 / 4} = 4.97 \text{ m/s} \]

\[ \vec{r} \times \vec{F}_s = 0 \] (Since atmospheric pressure acts on the entire control surface, and the pressure force at the inlet causes no moment about O.)
\( \int \vec{r} \times \vec{g} \rho \, dV = 0 \) (The moments of the body gravity) forces in the two arms are equal and opposite and hence the second term on the left side of the equation is zero.)

\( \vec{T}_{\text{shaft}} = -T_f \vec{K} \) (The only external torque acting on the CV is friction in the pivot. It opposes the motion.)

\( \frac{\partial}{\partial t} \int \vec{r} \times \vec{V} \rho \, dV = 0 \) (Because the sprinkler rotates at constant speed the control volume angular momentum is constant in XYZ coordinates, so this term is zero.)

To develop expressions for the instantaneous position vector, \( \vec{r} \), and velocity vector, \( \vec{V} \) of each element of fluid in the control volume.
We assume that the length, $L$, of the tip $AB$ is small compared with the length, $R$, of the horizontal arm $OA$. Consequently we neglect the angular momentum of the fluid in the tips compared with the angular momentum in the horizontal arms.
\[ \vec{V} = (V_\theta \cos \theta - r \omega \sin \theta) \vec{I} + (V_\theta \sin \theta + r \omega \cos \theta) \vec{J} \]
\[ \vec{r} = r \cos \theta \vec{I} + r \sin \theta \vec{J} \]
\[ \vec{r} \times \vec{V} = \left( r^2 \omega \cos^2 \theta + r^2 \omega \sin^2 \theta \right) \vec{K} = r^2 \omega \vec{K} \]
\[ \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho d\mathbf{V} = \frac{\partial}{\partial t} \int_0^R r^2 \omega \vec{K} \rho Adr = \frac{\partial}{\partial t} \left( \frac{R^3 \omega}{3} \vec{K} \rho A \right) = 0 \]

\( A \): the cross-sectional area of the horizontal tube.

There are three surfaces through which we have mass and therefore momentum flux: the supply line (for which \( \vec{r} \times \vec{V} = 0 \)) because \( \vec{r} = 0 \) and the two nozzles. Consider the nozzle at the end of branch \( OAB \). For \( L << R \), we have
\[ \vec{r}_{jet} = \vec{r}_B \approx \vec{r}_{|r=R} = R \cos \theta \vec{I} + R \sin \theta \vec{J} \]
\[ \vec{V}_j = \vec{V}_{\text{rel}} + \vec{V}_{\text{tip}} = \left( V_{\text{rel}} \cos \alpha \sin \theta \vec{I} - V_{\text{rel}} \cos \alpha \cos \theta \vec{J} + V_{\text{rel}} \sin \alpha \vec{K} \right) \]
\[ + \left( -R \omega \sin \theta \vec{I} + R \omega \cos \theta \vec{J} \right) \]
\[ = \left( V_{\text{rel}} \cos \alpha - R \omega \right) \sin \theta \vec{I} + \left( R \omega - V_{\text{rel}} \cos \alpha \right) \cos \theta \vec{J} + V_{\text{rel}} \sin \alpha \vec{K} \]
\[ \vec{r}_B \times \vec{V}_j = \left( R \cos \theta \vec{I} + R \sin \theta \vec{J} \right) \]
\[ \times \left[ \left( V_{\text{rel}} \cos \alpha - R \omega \right) \sin \theta \vec{I} + \left( R \omega - V_{\text{rel}} \cos \alpha \right) \cos \theta \vec{J} + V_{\text{rel}} \sin \alpha \vec{K} \right] \]
\[ = RV_{\text{rel}} \sin \alpha \sin \theta \vec{I} - RV_{\text{rel}} \sin \alpha \cos \theta \vec{J} \]
\[ - R \left( V_{\text{rel}} \cos \alpha - R \omega \right) \left( \sin^2 \theta + \cos^2 \theta \right) \vec{K} \]
\[ = RV_{\text{rel}} \sin \alpha \sin \theta \vec{I} - RV_{\text{rel}} \sin \alpha \cos \theta \vec{J} - R \left( V_{\text{rel}} \cos \alpha - R \omega \right) \vec{K} \]
The velocity and radius vectors for flow in the left arm must be described in terms of the same unit vectors used for the right arm. In the left arm the $\vec{I}$ and $\vec{J}$ components of the cross product are of opposite sign, since $\sin(\theta+\pi)=-\sin(\theta)$ and $\cos(\theta+\pi)=-\cos(\theta)$.
\[ T_f \ddot{K} = R \left( V_{rel} \cos \alpha - R \omega \right) \rho Q \ddot{K} \]

\[ T_f = R \left( V_{rel} \cos \alpha - R \omega \right) \rho Q \]

\[ = \frac{150}{1000} \left[ (4.97)(\cos 30^\circ) - \frac{150}{1000} \frac{(30)2\pi}{60} \right] (999) \left( \frac{7.5}{(1000)(60)} \right) \]

\[ = 0.0718 \, N \cdot m \]

4.8 The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy.

\[ \left( \frac{dE}{dt} \right)_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV + \int_{CS} e \rho \vec{V} \cdot d\vec{A} \]
\[
\dot{Q} - \dot{W} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV + \int_{CS} e \rho \vec{V} \cdot d\vec{A} \tag{4.54}
\]

where \( e = u + \frac{V^2}{2} + gz \).

Is Eq. 4.54 the form of the first law used in thermodynamics? Even for steady flow, Eq. 4.54 is not quite the same form used in applying the first law to control volume problems.

**Rate of Work Done by a Control Volume:**

- The term \( \dot{W} \) in Eq. 4.54 has a positive numerical value when work is done by the control volume on the surroundings.
- The rate of work done by the control volume is conveniently subdivided into four classifications,

\[
\dot{W} = \dot{W}_s + \dot{W}_{\text{normal}} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}
\]
Shaft Work: We shall designate shaft work $W_s$ and hence the rate of work transferred out through the control surface by shaft work is designated $\dot{W}_s$. Examples of shaft work are the work produced by the steam turbine (positive shaft work) of a power plant, and the work input required to run the compressor of a refrigerator (negative shaft work).

Work Done by Normal Stresses at the Control Surface: Recall that work requires a force to act through a distance. Thus, when a force, $\vec{F}$, acts through an infinitesimal displacement, $d\vec{s}$, the work done is given by $\delta W = \vec{F} \cdot d\vec{s}$. Thus the rate of work done by the force is

$$\dot{W} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F} \cdot d\vec{s}}{\Delta t} \quad \text{or} \quad \dot{W} = \vec{F} \cdot \vec{V}$$
Consider the segment of control surface shown in Fig. 4.6. For an element of area $d\vec{A}$ we can write an expression for the normal stress force $d\vec{F}_{\text{normal}}$: It will be given by the normal stress $\sigma_{nn}$ multiplied by the vector area element $d\vec{A}$ (normal to the control surface).

Hence the rate of work done on the area element is

$$d\vec{F}_{\text{normal}} \cdot \vec{V} = \sigma_{nn} d\vec{A} \cdot \vec{V}$$
Since the work out across the boundaries of the control volume is the negative of the work done on the control volume, the total rate of work out of the control volume due to normal stresses is

\[ \dot{W}_{\text{normal}} = -\int_{CS} \sigma_{nn} \, d\vec{A} \cdot \vec{V} = -\int_{CS} \sigma_{nn} \vec{V} \cdot d\vec{A} \]

**Work Done by Shear Stresses at the Control Surface:** The shear force acting on an element of area of the control surface is given by

\[ d\vec{F}_{\text{shear}} = \vec{\tau} \, dA \]

The rate of work done on the entire control surface by shear stresses is given by
\[ \dot{W}_{\text{shear}} = - \int_{CS} \vec{\tau} \cdot \vec{V} \, dA \]

- This integral is better expressed as three terms

\[ \dot{W}_{\text{shear}} = - \int_{CS} \vec{\tau} \cdot \vec{V} \, dA = - \int_{A(\text{shafts})} \vec{\tau} \cdot \vec{V} \, dA - \int_{A(\text{solid surface})} \vec{\tau} \cdot \vec{V} \, dA - \int_{A(\text{ports})} \vec{\tau} \cdot \vec{V} \, dA \]

- We have already accounted for the first term, since we included \( \dot{W}_s \) previously. At solid surfaces, \( \vec{V} = 0 \), so the second term is zero (for a fixed control volume). Thus,

\[ \dot{W}_{\text{shear}} = - \int_{A(\text{ports})} \vec{\tau} \cdot \vec{V} \, dA \]

- This last term can be made zero by proper choice of control
surfaces. If we choose a control surface that cuts across each port perpendicular to the flow, then \( d\vec{A} \) is parallel to \( \vec{V} \). Since \( \vec{\tau} \) is in the plane of \( d\vec{A} \), \( \vec{\tau} \) is perpendicular to \( \vec{V} \). Thus, for a control surface perpendicular to \( \vec{V} \),

\[
\vec{\tau} \cdot \vec{V} = 0 \quad \text{and} \quad \dot{W}_{\text{shear}} = 0
\]

**Other Work:** Electrical energy could be added to the control volume. Also electromagnetic energy, e.g., in radar or laser beams, could be absorbed. In most problems, such contributions will be absent, but we should note them in our general formulation.

With all of the terms in \( \dot{W} \) evaluated, we obtain

\[
\dot{W} = \dot{W}_s - \int_{CS} \sigma_{nn} \vec{V} \cdot d\vec{A} + \dot{W}_{\text{shear}} + \dot{W}_{\text{other}}
\]
Control Volume Equation:

\[ \dot{Q} - \dot{W}_s + \int_{CS} \sigma_{nn} \vec{V} \cdot d\vec{A} - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV + \int_{CS} e \rho \vec{V} \cdot d\vec{A} \]

Since \( \sigma_{nn} \approx -p \), \( e = u + \frac{V^2}{2} + gz \)

\[ \dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t} \int_{CV} e \rho \, dV + \int_{CS} \left( u + pv + \frac{V^2}{2} + gz \right) \rho \vec{V} \cdot d\vec{A} \quad (4.56) \]

Each work term in Eq. 4.56 represents the rate of work done by the control volume on the surroundings.

Note that in thermodynamics, for convenience, the combination \( u + pv \) (the fluid internal energy plus what is often called the "flow work") is usually replaced with enthalpy, \( h \equiv u + pv \) (this is one of the reasons \( h \) was invented).

Ex. 4.16 Compressor- First Law Analysis: Air at 101 kPa, 21°C,
enters a compressor with negligible velocity and is discharged at 344 kPa, 38 °C through a pipe with 0.09 m² area. The flow rate is 9 kg/s. The power input to the compressor is 447 kW. Determine the rate of heat transfer.

**Given:** Air enters a compressor at 1 and leaves at 2 with conditions as shown. The air flow rate is 9 kg/s and the power input to the compressor is 447 kW.

**Find:** Rate of heat transfer.

**Solution:**
G.E. \[ \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d \vec{A} = 0 \]

\[ \dot{Q} - \dot{W_s} - \dot{W_{shear}} = \frac{\partial}{\partial t} \int_{CV} e \rho dV + \int_{CS} \left( u + pv + \frac{V^2}{2} + gz \right) \rho \vec{V} \cdot d \vec{A} \]

Assumptions: (1) Steady flow; (2) Properties uniform over inlet and outlet sections; (3) Treat air as an ideal gas, \( p = \rho RT \); (4) Area of CV at 1 and 2 perpendicular to velocity, thus \( \dot{W}_{shear} = 0 \); (5) \( z_1 = z_2 \); (6) Inlet kinetic energy is negligible.
\[ \dot{Q} - \dot{W}_s = \sum_{\text{CS}} \left( u + p v + \frac{V^2}{2} + g z \right) \rho \vec{V} \cdot \vec{A} = \sum_{\text{CS}} \left( h + \frac{V^2}{2} + g z \right) \rho \vec{V} \cdot \vec{A} \]

\[ = \left( h_1 + \frac{V_1^2}{2} + g z_1 \right) (-\rho V_1 A_1) + \left( h_2 + \frac{V_2^2}{2} + g z_2 \right) (\rho V_2 A_2) \]

\[ \sum_{\text{CS}} \rho \vec{V} \cdot \vec{A} = 0 \Rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m} \]

\[ \dot{Q} - \dot{W}_s = \left[ (h_2 - h_1) + \frac{1}{2} \left( V_2^2 - V_1^2 \right) + g (z_2 - z_1) \right] \dot{m} \]

\[ \Rightarrow \dot{Q} = \dot{W}_s + \left[ (h_2 - h_1) + \frac{V_2^2}{2} \right] \dot{m} = \dot{W}_s + \left[ c_p (T_2 - T_1) + \frac{V_2^2}{2} \right] \dot{m} \]

\[ V_2 = \frac{\dot{m}}{\rho_2 A_2} = \frac{RT \dot{m}}{p_2 A_2} = \frac{(287)(273+38)}{344 \times 10^5} \frac{9}{0.09} = 25.9 \text{ m/s} \]
\[ \dot{Q} = \dot{W}_s + \left[ c_p (T_2 - T_1) + \frac{V_2^2}{2} \right] \dot{m} \]

\[ = (-447 \times 10^3) + \left[ 1005(38 - 21) + \frac{25.9^2}{2} \right] \]

\[ = -290.2 \times 10^3 \text{ W} = -290.2 \text{ kW} \]

**Ex. 4.17 Tank Filling- First Law Analysis:** A tank of 0.1 m$^3$ volume is connected to a high-pressure air line; both line and tank are initially at a uniform temperature of 20°C. The initial tank gage pressure is 100 kPa. The absolute line pressure is 2.0 MPa; the line is large enough so that its temperature and pressure may be assumed constant. The tank temperature is monitored by a fast-response thermocouple. At the instant after the valve is opened,
the tank temperature rises at the rate of 0.05°C/s. Determine the instantaneous flow rate of air into the tank if heat transfer is neglected.

**Given:** Air supply pipe and tank as shown. At \( t=0 \), \( \partial T/\partial t = 0.05 \)°C/s.

**Find:** \( \dot{m} \) at \( t=0 \).

**Solution:**

\[
\text{G.E.} \quad \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0
\]
\[ \dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} \]

\[ = \frac{\partial}{\partial t} \int_{CV} \left( u + \frac{V^2}{2} + gz \right) \rho dV + \int_{CS} \left( u + pv + \frac{V^2}{2} + gz \right) \rho \vec{V}_{xyz} \cdot d\vec{A} \]

Assumptions: (1) \( \dot{Q} = 0 \) (given); (2) \( \dot{W}_s = 0 \); (3) \( \dot{W}_{\text{shear}} = 0 \); (4) \( \dot{W}_{\text{other}} = 0 \); (5) Velocities in line and tank are small; (6) Neglect potential energy; (7) Uniform flow at tank inlet; (8) Properties uniform in tank; (9) Ideal gas, \( p = \rho RT \), \( du = c_v dT \).
\[
\frac{\partial}{\partial t} \int_{CV} u_{\text{tank}} \rho dV + (u + pv)_{\text{line}} (-\rho VA) = 0
\]

\[
\Rightarrow \frac{d(uM)}{dt} + (u + RT)(-\rho VA) = 0
\]

\[
\Rightarrow \frac{d(uM)}{dt} + (u + RT)(-\dot{m}) = 0
\]

\( (T \text{ is uniform at } 20^\circ C \Rightarrow u_{\text{tank}} = u_{\text{line}} = u) \)

Since tank properties are uniform, \( \partial / \partial t \) may be replaced by \( d/dt \).

\[
\frac{\partial}{\partial t} \int_{CV} \rho dV + \sum_{CS} \rho \vec{v} \cdot \vec{A} = 0 \Rightarrow \frac{dM}{dt} + (-\rho VA) = 0
\]

\[
\Rightarrow \frac{dM}{dt} = \rho VA = \dot{m}
\]
\[
\Rightarrow \frac{d(uM)}{dt} = (u + RT)\dot{m} \Rightarrow u \frac{dM}{dt} + M \frac{du}{dt} = (u + RT)\dot{m}
\]

\[
\Rightarrow um + M \frac{du}{dt} = um + RT\dot{m} \Rightarrow \dot{m} = \frac{M}{RT} \frac{du}{dt} = \frac{\rho \mathcal{V}}{RT} c_v \frac{dT}{dt}
\]

\[
\dot{m} = \frac{\rho \mathcal{V}}{RT} c_v \frac{dT}{dt} = \frac{p}{RT} \frac{\mathcal{V}}{RT} c_v \frac{dT}{dt}
\]

\[
= \left(1 \times 10^5 + 1.01 \times 10^5 \right) \frac{0.1}{(287)(273 + 20)} \frac{717}{(273 + 20)} \times (0.05)
\]

\[
= 0.102 \times 10^{-3} \text{ kg/s} = 0.102 \text{ g/s}
\]

4.9 The Second Law of Thermodynamics

- To derive the control volume formulation of the second law of
thermodynamics, we obtain

\[
\frac{dS}{dt}_{\text{system}} \geq \frac{1}{T} \dot{Q}
\]

\[
\frac{dS}{dt}_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} s \rho \, dV + \int_{CS} s \rho \vec{V} \cdot d\vec{A}
\]

\[
\frac{\partial}{\partial t} \int_{CV} s \rho \, dV + \int_{CS} s \rho \vec{V} \cdot d\vec{A} \geq \int_{CS} \frac{1}{T} \left( \frac{\dot{Q}}{A} \right) dA \quad (4.58)
\]

- In Eq. 4.58, the factor \((\dot{Q} / A)\) represents the heat flux per unit area into the control volume through the area element \(dA\). To evaluate the term

\[
\int_{CS} \frac{1}{T} \left( \frac{\dot{Q}}{A} \right) dA
\]
both the local heat flux, \((\dot{Q}/A)\), and local temperature, \(T\), must be known for each area element of the control surface.

4.10 Summary and Useful Equations

In this chapter we wrote the basic laws for a system: mass conservation (or continuity), Newton’s second law, the angular momentum equation, the first law of thermodynamics, and the second law of thermodynamics. We then developed an equation (sometimes called the Reynolds Transport Theorem) for relating system formulations to control volume formulations. Using this we derived control volume forms of:

- The mass conservation equation (sometimes called the continuity equation).
Newton’s second law (in other words, a momentum equation) for:

- An inertial control volume.
- A control volume with rectilinear acceleration.
- A control volume with arbitrary acceleration (on the Web).

The angular-momentum equation for:

- A fixed control volume.
- A rotating control volume (on the Web).

The first law of thermodynamics (or energy equation).

The second law of thermodynamics.
## Useful Equations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4.13a)</td>
<td>Continuity (mass conservation), incompressible fluid: $\int_{CS} \vec{V} \cdot d\vec{A} = 0$</td>
<td>Page 105</td>
</tr>
<tr>
<td>(4.13b)</td>
<td>Continuity (mass conservation), incompressible fluid, uniform flow: $\sum_{CS} \vec{V} \cdot \vec{A} = 0$</td>
<td>Page 105</td>
</tr>
<tr>
<td>(4.15a)</td>
<td>Continuity (mass conservation), steady flow: $\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$</td>
<td>Page 106</td>
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<td>Page 106</td>
</tr>
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<td>(4.17a)</td>
<td>Momentum (Newton’s second law): $\vec{F} = \vec{F}<em>S + \vec{F}<em>B = \frac{\partial}{\partial t} \int</em>{CV} \vec{V} \rho d\vec{V} + \int</em>{CS} \vec{V} \rho \vec{V} \cdot d\vec{A}$</td>
<td>Page 111</td>
</tr>
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<td>Page 111</td>
</tr>
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<td>Page</td>
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<tr>
<td>$F_x = F_{Sx} + F_{Bx} = \frac{\partial}{\partial t} \int_{CV} u \rho , dV + \int_{CS} u \rho \vec{V} \cdot d\vec{A}$</td>
<td>(4.18a)</td>
<td>112</td>
</tr>
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<td>$F_y = F_{Sy} + F_{By} = \frac{\partial}{\partial t} \int_{CV} v \rho , dV + \int_{CS} v \rho \vec{V} \cdot d\vec{A}$</td>
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<td>$F_z = F_{Sz} + F_{Bz} = \frac{\partial}{\partial t} \int_{CV} w \rho , dV + \int_{CS} w \rho \vec{V} \cdot d\vec{A}$</td>
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<td>112</td>
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<tr>
<td>$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$</td>
<td>(4.24)</td>
<td>124</td>
</tr>
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<td>$\vec{F} = \vec{F}<em>S + \vec{F}<em>B = \frac{\partial}{\partial t} \int</em>{CV} \vec{V}</em>{xyz} \rho , dV + \int_{CS} \vec{V}<em>{xyz} \rho \vec{V}</em>{xyz} \cdot d\vec{A}$</td>
<td>(4.26)</td>
<td>126</td>
</tr>
<tr>
<td>$\vec{F}<em>S + \vec{F}<em>B - \int</em>{CV} \vec{a}</em>{rf} \rho , dV = \frac{\partial}{\partial t} \int_{CV} \vec{V}<em>{xyz} \rho , dV \int</em>{CS} \vec{V}<em>{xyz} \rho \vec{V}</em>{xyz} \cdot d\vec{A}$</td>
<td>(4.33)</td>
<td>130</td>
</tr>
<tr>
<td>Angular-momentum principle: $\vec{r} \times \vec{F}<em>S + \int</em>{CV} \vec{r} \times \vec{g} \rho , dV + \vec{T}_{\text{shaft}}$</td>
<td>(4.46)</td>
<td>136</td>
</tr>
<tr>
<td>$= \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho , dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$</td>
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