Turbulent Transport of Momentum and Heat

- Turbulent consists of random velocity fluctuations, so that it must be treated with statistical methods.
- The statistical analysis decomposes all quantities into mean value and fluctuations with zero mean.
- Turbulent velocity fluctuations can generate large momentum fluxes, which can be thought of as a stress or are commonly called Reynolds stress, between different parts of a flow.
- The momentum exchange mechanism superficially resembles molecular transport of momentum, and by analogy, this exchange process is often represented by an eddy viscosity.
The Reynolds Equations

• In turbulence, a description of the flow at all points in time and space is not feasible.

• Instead, following Reynolds, we develop equations governing mean quantities, such as mean velocity.

• The equation of motion of an incompressible fluid are,

\[
\frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial \tilde{u}_1}{\partial x_1} + \frac{\partial \tilde{u}_2}{\partial x_2} + \frac{\partial \tilde{u}_3}{\partial x_3}, \quad \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j}
\]

where \( \tilde{\sigma}_{ij} \) is the stress tensor. Repeated indices in any term indicate a summation over all three values of the index; a tilde denotes the instantaneous value at \((x_i, t)\) of a variable.
• If the fluid is Newtonian, the stress tensor is given by, \( \tilde{\sigma}_{ij} = -\tilde{p} + 2\mu\tilde{s}_{ij} \)
the rate of strain is defined by, \( \tilde{s}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) \)
• The final form of the Navier-Stokes equation can be expressed as,

\[
\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}
\]
The Reynolds Decomposition

- The instantaneous velocity is decomposed into a mean flow and velocity fluctuations,
  \[ \tilde{u}_i = U_i + u_i \]
- \( U_i \) is defined, in terms of time average, as,
  \[ \bar{u}_i = U_i = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \tilde{u}_i \, dt \]
- Time averages (mean values) of fluctuations and of their derivatives, products and other combinations are denoted by an overbar.
- The mean value of a fluctuating quantity is zero by definition;
  \[ \overline{u}_i = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} (\tilde{u}_i - U_i) \, dt = 0 \]
• The use of time averages corresponds to the typical laboratory situations in which measurements are taken at fixed locations in a stationary, steady, but often inhomogeneous flow field.

• For a time average to make sense, the mean flow has to be steady,
  \[ \frac{\partial \bar{U}_i}{\partial t} = 0 \]

• The mean values of spatial derivative of a variable is equal to the corresponding spatial derivative of the mean value of that variable.
  \[ \frac{\partial \bar{u}_i}{\partial x_j} = \frac{\partial \bar{U}_i}{\partial x_j}, \quad \bar{u}_i = \bar{u}_i = 0 \]
• The pressure and the stress are also decomposed into mean and fluctuating components.

\[ \tilde{p} = P + p, \quad \bar{p} = 0 \]

\[ \tilde{\sigma}_{ij} = \Sigma_{ij} + \sigma_{ij}, \quad \bar{\sigma}_{ij} = 0 \]

and, \[ \Sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij}, \quad \sigma_{ij} = -p\delta_{ij} + 2\mu s_{ij} \]

• The mean strain rate and the strain-rate fluctuations are defined as,

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
Correlated Variables

• Averages of products are computed in the following way,

\[
\tilde{u}_i \tilde{u}_j = (U_i + u_i) \times (U_j + u_j) \\
= U_i U_j + \overline{u_i u_j} + \overline{U_i u_j} + \overline{u_i U_j} = U_i U_j + \overline{u_i u_j}
\]

• If \( \overline{u_i u_j} \neq 0 \), \( u_i \) and \( u_j \) are said to be correlated;
if \( \overline{u_i u_j} = 0 \), the two are uncorrelated. (see Fig. 2.1)

• A measure of the degree of correlation between the two variables \( u_i \) and \( u_j \) is the correlation coefficient \( c_{ij} \),

\[
c_{ij} = \frac{u_i u_j}{\sqrt{\left(\overline{u_i^2} \overline{u_j^2}\right)}^{1/2}}
\]

• Where the summation convention does not apply in this case.
• If \( c_{ij} = \pm 1 \), the correlation is said to be perfect. Each variable, of course, is perfectly correlated with itself. (\( c_{\alpha\alpha} = 1 \), if \( i = j = \alpha \))

• The square root of a variance is called a standard deviation or root-mean-square;

\[
u_i' = \left( \frac{u_i^2}{u_i} \right)^{1/2}
\]

which is a characteristic velocity or velocity scale of turbulence.
Equations for the Mean Flow

- If the Reynolds decomposition is applied to continuity equation for incompressible fluid, it becomes,
  \[
  \frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial (U_i + u_i)}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0
  \]

- If the average of all terms is taken, the last term vanishes, then
  \[
  \frac{\partial U_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x_i} = 0
  \]

- The equations of motion for the mean flow $U_i$ are obtained as followed,
  \[
  \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j}
  \]
Using the Reynolds decomposition and taking the average of all terms in the resulting equation,

\[
\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j) \frac{\partial (U_i + u_i)}{\partial x_j} = \frac{1}{\rho} \frac{\partial (\Sigma_{ij} + \sigma_{ij})}{\partial x_j}
\]

- This yields,

\[
U_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial \Sigma_{ij}}{\partial x_j}
\]

- With the use of the continuity equation for the turbulent velocity fluctuations, we may write,

\[
u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i u_j}{\partial x_j}
\]

This represents the mean transport of fluctuating momentum by turbulent velocity fluctuations. If \(u_i\) and \(u_j\) were uncorrelated, there would be no turbulent momentum transfer.
Experience shows that momentum transfer is a key feature of turbulent motion; and this term may change the momentum of the mean flow by exchanging momentum between the turbulence and the mean flow.

Because of the Reynolds decomposition, the turbulent motion can be perceived as an agency that produces stress in the mean flow.

Thus the equation of motion can be rearranged as, “Reynolds momentum equation”,

\[
U_j \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \Sigma_{ij} - \rho \bar{u}_i \bar{u}_j \right)
\]

where \( \Sigma_{ij} = -P \delta_{ij} + 2 \mu S_{ij} \), \( S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \)
The Reynolds Stress

- The contribution of the turbulent motion to the mean stress tensor is designated by the symbol $\tau_{ij}$,
  
  $\tau_{ij} = -\rho u_i u_j$

  where $\tau_{ij}$ is called the Reynolds stress tensor, in honor of the original developer of this part of the theory.

- The Reynolds stress is symmetric.

- The diagonal components are normal stress,
  
  $-\rho u_1^2, -\rho u_2^2, \text{ and } -\rho u_3^2$,

  and in many flows these normal stresses contribute little to the transport of mean momentum.

- The off diagonal components of $\tau_{ij}$ are shear stresses; they play a dominant role in the theory of mean momentum transfer by turbulent motion.
The decomposition of the flow into a mean and turbulent velocity fluctuations has isolated the effects of fluctuations on the mean flows.

However, the mean flow equations contain additional unknowns $\tau_{ij}$ and this illustrates the closure problem of turbulence.

Indeed, if one obtains additional equations for $\tau_{ij}$ from the original Navier-Stokes equations, unknowns like $u_i u_j u_k$ are generated by the nonlinear inertia terms.

Many investigators have attempted to guess at a relation between $\tau_{ij}$ and $S_{ij}$; because the functions of the Reynolds stress in the equation of motion seems to be similar to that of the viscous stress $2\mu S_{ij}$.

$$\frac{\partial U_i U_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \rho u_i u_j \right]$$
Turbulent Transport of Heat

- Turbulence transports passive contaminants such as heat, chemical species and particles in much the same way as momentum.
- The density is approximately constant if temperature differences remain relatively small, if gravity induced density stratification may be neglected, and if Mach number of the flow is small.
- The instantaneous diffusion equation for heat in a flow is,

\[
\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\theta}}{\partial x_j} = \gamma \frac{\partial^2 \tilde{\theta}}{\partial x_j \partial x_j}
\]

where \( \gamma \), thermal diffusivity \( \text{m}^2/\text{s} \), is assumed to be constant.
• The temperature $\tilde{\theta}$ at $(x_i, t)$ is decomposed into a mean value $\Theta$ and a temperature fluctuation $\theta$.

$$\tilde{\theta} = \Theta + \theta$$

$$\bar{\theta} = \Theta = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \theta dt$$

$$\bar{\theta} = 0, \quad \frac{\partial \Theta}{\partial t} = 0$$

• Thus, the mean diffusion equation for heat in a flow is,

$$\frac{\partial U_j \Theta}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \gamma \frac{\partial \Theta}{\partial x_j} - \bar{\theta} \dot{u}_j \right)$$
The mean heat flux $Q_j$ per unit area and unit time in a turbulent flow then becomes,

$$Q_j = C_p \rho \left( \overline{\theta u_j} - \gamma \frac{\partial \Theta}{\partial x_j} \right)$$

which is a sum of the contributions of the molecular motion and of the turbulent motion.

The similarity between the mean stress tensor and the heat flux tensor is the analytical foundation for the belief that turbulence may transport heat in much the same way as momentum.
Elements of the Kinetic Theory of Gases

- The molecular background of the viscosity and other molecular transport coefficients in dilute perfect gases will be introduced.
- For gases, the rudiments of kinetic theory are straightforward, but the kinetic theory of liquids is not nearly as well developed.
Pure Shear Flow

• Consider a steady pure shear flow, where $U_1 = f(x_2)$ and $U_2 = U_3 = 0$. If the flow is laminar, the only nonvanishing components of the viscous shear stress are, $\sigma_{12} = \sigma_{21} = \mu \frac{\partial U_1}{\partial x_2}$

• The shear stress $\sigma_{12}$ must result from molecular transport of momentum in the $x_2$ direction only.

• Let $v_1$ and $v_2$ be the $x_1$ and $x_2$ components of the instantaneous velocity of a molecular relative to the mean flow.

• The $x_1$ momentum $mv_1$ of a molecule with mass $m$ is transported in the $x_2$ direction if $v_2$ is correlated with $v_1$. 
• The momentum transport per molecule is proportional to $mv_1 v_2$. If there are $N$ molecules per unit volume, the transport of $x_1$ momentum in the $x_2$ direction is $Nm v_1 v_2$ per unit time and area.

• Momentum flux per unit area and time may be equated with a stress

$$\sigma_{12} = -\rho v_1 v_2$$

• The minus sign is needed because positive values of $v_2$ should carry momentum deficit in a flow with positive $\sigma_{12}$ and $\partial U_1/\partial x_2$. 
Molecular Collisions

- A molecule coming from $x_2=-\xi$ collides with another molecule at the reference level $x_2=0$. (Fig. 2.3)
- Assuming that because of this collision the molecule coming from below adjusts its momentum in the $x_1$ direction to that of its new environment, it has to absorb an amount of momentum equal to $M=m[U_1(0)-U_1(-\xi)]$ which is equal to the amount of momentum lost by the environment at $x_2=0$.
- $M$ may be expanded in a Taylor series, $M=m\xi \partial U_1/\partial x_2 + 0.5m\xi^2 \partial^2 U_1/\partial x_2^2 + \ldots$.
- The second and higher terms may be neglected if, $\partial U_1/\partial x_2 >> 0.5 \xi \partial^2 U_1/\partial x_2^2$. 
• A local length scale $l$ of the flow $U_1(x_2)$ is defined as, $l = (\partial U_1/\partial x_2)/(\partial^2 U_1/\partial x_2^2)$.
• Adopting the definition of length scale, the first and second order derivative inequality can be written as, $l \gg 0.5 \xi$.
• For air at room temperature and density, $\xi = 7 \times 10^{-8}$ m, so that for almost all flows the condition is satisfied.
• Thus, $M$ can be approximated by, $M = m \xi \partial U_1/\partial x_2$.
• The number of collisions occurring at the reference level $x_2 = 0$ per unit area and time may be estimated as $Na$.
• If the momentum transfer per collision is $M$, the momentum transfer per unit area and time must be proportional to $MNa$. 
• Thus the stress can be written as,
\[ \sigma_{12} = \alpha M Na = \alpha N ma \xi \partial U_1 / \partial x_2 \]
where \( \alpha \) is an unknown coefficient, which should be of order one.

• In air at ordinary temperature and pressure, \( \alpha \) is approximately 2/3. Therefore
\[ \sigma_{12} = (2/3) \rho a \xi \partial U_1 / \partial x_2 = \mu \partial U_1 / \partial x_2, \quad v = (2/3)a \xi \]

• The Re formed with these variables is, \( a \xi / v = 3/2 \). That this Re turns out to be of the order one is no accident because the viscosity is defined on the basis of molecular motion with velocity \( a \) and length scale \( \xi \).

e.g. air at room temperature and pressure, \( \xi = 7 \times 10^{-8} \) m, \( a = 3.4 \times 10^2 \) m/s then \( v = 15 \times 10^{-6} \) m\(^2\)/s.
Characteristic Times and Lengths

- Knudsen number $K$: ratio of $\xi$ to the local length scale $l$ of the flow
  \[ K = \frac{\xi}{l} = \frac{3}{2} \frac{v/a}{(U/a)(v/Ul)} = \frac{3}{2} \frac{M}{R} \]

- In most flows, $M \ll R$, so that $l \gg 1/2 \xi$ is easily satisfied.

- Apart from the ratio of length scales, the time scales involved in momentum transport of momentum are of interest.

- The molecular time scale is the time interval $\xi/a$ between collisions, $10^{-10}$ s. The time scale of the flow is the reciprocal of the velocity gradient $\partial U_1/\partial x_2$, for a quite rapid shearing flow, the velocity gradient is $10^4$ s$^{-1}$ and the time scale of the flow is $10^{-4}$ s.

- Changes in the flow are slow compared to the time scale representing molecular motion.

- This suggests that the thermal motion of the molecules should not be disturbed very much by the flow.
The Correlation between $v_1$ and $v_2$

- $\sigma_{12} = \alpha N m a \xi \partial U_1 / \partial x_2 = -\rho v_1 v_2$

- The part of $v_1$ correlated with $v_2$ is proportional to $\xi \partial U_1 / \partial x_2$.

- For a rapid flow in air ($\xi = 7 \times 10^{-8}$ m, $\partial U_1 / \partial x_2 = 10^4$ s$^{-1}$), $\xi \partial U_1 / \partial x_2 = 7 \times 10^{-4}$ m/s.

- A correlation coefficient $c$ between $v_1$ and $v_2$ may be defined as,

$$c = -\frac{v_1 v_2}{(v'_2)^2}$$

where $v'_2$ is the rms value of the $x_2$ component of the molecular velocity which is of the same order of magnitudes as the speed of sound $a$, $3.4 \times 10^2$ m/s for air at room temperature, and $v'_1 = v'_2$ is adopted.
Thus the correlation coefficient $c$ may be estimated as,

$$c \sim \frac{\xi \partial U_1 / \partial x_2}{v'_2} = 2 \times 10^{-6}$$

This indicates that $v_1$ and $v_2$ are very poorly correlated.

If $\partial U_1/\partial x_2$ is estimated as $U/l$, then

$$c \sim \frac{\xi \partial U_1 / \partial x_2}{v'_2} \sim \frac{\xi U / l}{a} = \frac{3}{2} \frac{v}{a^2} \frac{U}{l} = \frac{3 M^2}{2 R}$$

which tends to be extremely small in most flow.

We may conclude that the state of the gas is hardly disturbed by molecular momentum transfer.
• Dynamical equilibrium of the thermal motion of the molecules in shear flow of gases is, to a very close approximation, the same as the equilibrium state in a gas at rest.

• This implies that shear flow is not likely to upset the equation of state of the gas, unless $M^2/R$ is large.

• Hence, we may anticipate that the correlation coefficient of turbulent velocity fluctuations is not small in turbulent shear flow.

• Consequently, the state of turbulence is not independent of the mean flow field; on the contrary, the interaction between the mean flow and turbulence tends to be quite strong.
Thermal Diffusivity

- Molecular transport of scalar quantities is similar to the transport of momentum

\[ Q_2 = -\rho C_p \gamma \frac{\partial \Theta}{\partial x_2} = -0.93 C_p \rho a \xi \frac{\partial \Theta}{\partial x_2} \]

where, \( \nu = \frac{2}{3} a \xi \), \( \nu/\gamma = 0.73 \) for air at room temperature.

- The thermal diffusivity is larger than the diffusivity of momentum, because molecules that travel faster than average carry more thermal energy with them and make more collisions per unit time.

- Energetic molecules thus do more than a proportional share in transporting heat.
Estimates of the Reynolds Stress

- We have seen that molecular transport can be interpreted fairly easily in terms of the parameters of molecular motion.
- It is very tempting to apply a similar heuristic treatment to turbulent transport.
- We again use a turbulent pure shear flow as a basis for our discussion.
- Consider a pure shear flow, then the rates of turbulent momentum transfer and heat transfer are,

\[ \tau_{12} = -\rho u_1 u_2, \quad H_2 = \rho C_p \theta u_2 \]
Reynolds Stress and Vortex Stretching

- The existence of a Reynolds stress requires that the velocity fluctuations $u_1$ and $u_2$ be correlated.
- In a shear flow with $\partial U_1/\partial x_2 > 0$, negative value of $u_1$ should occur more frequently than positive ones when $u_2$ is positive and vice versa.
- The energy of the eddies has to be maintained by the shear flow, because they are continuously losing energy to smaller eddies.
- Molecules do not depend on the flow for their energy because the collisions between molecules are elastic.
- Eddies need shear to maintain their energy; the most powerful eddies thus are those that can absorb energy from shear flow more effectively than others.
- Evidence suggests that the eddies that are effective than most in maintaining the desired correlation between $u_1$ and $u_2$ and in extracting energy from the mean flow are vortices whose principal axis is roughly aligned with that of the mean strain rate.

- The energies transfer mechanism for eddies of this kind is believed to be associated with vortex stretching. (Fig. 2.5)

- As the eddies are being strained by the shear, conservation of angular momentum tends to maintain the good correlation between $u_1$ and $u_2$, thus allowing efficient energy transfer.

- The interaction between eddies and the mean flow is essentially three dimensional.
• Two dimensional eddies \( (u'_3 = 0) \) may on occasion have appreciable Reynolds stress, but the mean shear tends to rotate and strain them in such a way that they would lose their capacity for extracting energy from the mean flow rather quickly.

• The dynamic interaction between the mean flow and the turbulence is too strong to be represented by a simple transport theory model patterned after kinetic theory of gases which is at best a very crude representation of reality.

• A more detailed analysis of the energy and vorticity dynamics of the eddies is essential to the understanding of turbulence.
The Mixing Length Model

- For a moving point as it travels upward passing an arbitrary level $x_2$ at time $t$, it has a momentum deficit (Fig. 2.6),

$$\Delta M = \rho \tilde{u}_1(x_2, t) - \rho \tilde{u}_1(0, 0)$$

$$= \rho [U_1(x_2) - U_1(0)] + \rho [u_1(x_2, t) - u_1(0, 0)]$$

- If the contribution of the turbulence to the momentum deficit can be neglected, $\Delta M$ may be approximated by,

$$\Delta M = \rho x_2 \frac{\partial U_1}{\partial x_2}$$

where the gradient is taken at $x_2 = 0$.

- The volume transported per unit area and unit time in the $x_2$ direction is $\tilde{u}_2$ of the moving point, and $\tilde{u}_2 = dx_2 / dt$. 
The average momentum flux at $x_2$ may be written as,

$$\tau_{12} = \Delta \overline{M \ddot{u}_2} = \rho x_2 u_2 \frac{\partial U_1}{\partial x_2} = \rho \frac{\partial U_1}{2 \partial x_2} \frac{d}{dt} \left( \overline{x_2^2} \right)$$

The overbar denotes the average over all moving point that starts from $x_2$.

$$\frac{d}{dt} \left( \overline{x_2^2} \right) = 2x_2 \frac{dx_2}{dt} = 2\overline{x_2 u_2}$$

If the flow at any point did not continuously exchange momentum with its environment, $u_2$ would remain constant for any given moving point, and $x_2 u_2$ would continue to increase in time as $x_2$ increased.

This is not realistic; instead we expect that the correlation between $u_2$ and $x_2$ of a moving point decreases as the distance travel increases.
If we assume that \( u_2 \) and \( x_2 \) become essentially uncorrelated at values of \( x_2 \) comparable to some distance length scale, \( l \), then \[ x_2 u_2 \sim u'_2 l \]
where \( u'_2 \) is the rms velocity in the \( x_2 \) direction; the dispersion length scale \( l \) is called the mixing length.

Thus the stress can be estimated as,

\[
\tau_{12} = c_1 \rho u'_2 l \frac{\partial U_1}{\partial x_2} = \rho \nu_t \frac{\partial U_1}{\partial x_2}
\]

where \( \nu_t \) is the eddy viscosity (or turbulent exchange coefficient for momentum) and is given by

\[
\nu_t = c_1 u'_2 l
\]

If the mixing length and the velocity \( u'_2 \) were known everywhere in the flow field and if the mixing length model were accurate, the closure problem would be solved.
• However, $u'_2$ and $l$ are not properties of the fluid but properties of the flow and may vary throughout the flow field, making the eddy viscosity variable, dependent on the position in the flow.

• Consequently, application of mixing length model are usually restricted to flows for which $u'_2$ is approximately constant (at least in the cross stream direction) and for which $l$ is constant or depends on the geometry of the shear flow considered.

• The mixing length model favors large-scale motions which may be argued to contributed more to the momentum transfer than small eddies; and $l$ may be taken to be proportional to the size of the larger eddies.
The Length Scale Problem

- Define a local length scale $L$ of the mean flow by, $L = (\partial U_1 / \partial x_2) / (\partial^2 U_1 / \partial x_2^2)$.
- The approximation, $U_1(x_2) - U_1(0) = x_2 \partial U_1 / \partial x_2$ for all values of $x_2$ of order $l$ is valid if $L >> 0.5l$.
- In turbulent flow, however, the largest eddies tend to have sizes comparable to the width of the flow.
- Consequently, $l$ is usually of the same order as the local length scale $L$.
- This makes the turbulent Knudsen number $l/L$ of order one and the truncation of Taylor series expansion involved in determining the momentum deficit is not justified.
- Therefore, a gradient-transport model which links the stress to the rate of strain at the same point in time and space can not be used for turbulent flow.
The Mixing Length as an Integral Scale

* Since 
  \[ \frac{1}{2} \frac{d}{dt} \left( \overline{x_2^2} \right) = x_2 u_2 = c_1 u'_2 l \]
  it is worthwhile to investigate how \( l \) could be defined.

* Consider how the value of \( x_2 \) increases as the moving point travels away from the reference level at \( x_2=0 \),
  \[ x_2(t) = \int_0^t u_2(t') dt', \quad \frac{1}{2} \frac{d}{dt} \left( \overline{x_2^2} \right) = x_2 u_2 = \int_0^t u_2(t') u_2 dt'. \]
  the averaging process can be performed on the integrand because it is done over many moving points, not over time.

* In a statistically steady situation, the origin of time is irrelevant, so that the correlation between \( u_2(t) \) and \( u_2(t') \) should depend only on time difference \( t-t' = \tau \).
Define a correlation coefficient $c(\tau)$ by

$$c(\tau) = \frac{u_2(t)u_2(t-\tau)}{u_2^2}$$

thus,

$$\frac{1}{2} \frac{d}{dt} \left( \frac{x_2^2}{2} \right) = u_2^2 \int_0^t c(\tau)d\tau$$

At large values of $\tau$ the velocities $u_2(t)$ and $u_2(t')$ are uncorrelated. (Fig. 2.7)

The area under the curve in Fig. 2.7 is given by

$$\mathcal{T} = \int_0^\infty c(\tau)d\tau$$

where the time is called the Lagrangian integral (time) scale. It is also assumed that $c(\tau)$ decreases rapidly enough at large $\tau$ to make $\mathcal{T}$ finite.

Moving fluid loses its capability of transporting momentum when the correlation between $x_2$ and $u_2$ becomes zero.

The dispersion rate then becomes,

$$\frac{1}{2} \frac{d}{dt} \left( \frac{x_2^2}{2} \right) = u_2^2 \mathcal{T}$$
• Define a Lagrangian integral length scale $l_L$, 

\[ l_L = u'_2 \mathcal{T} \text{ then } \frac{1}{2} \frac{d}{dt} \left( \overline{x_2^2} \right) = u'_2 l_L \]

• The time scale $\mathcal{T}$ is hard to determine experimentally, because it requires that motion of many tagged fluid particles be followed.

• In most turbulent flow, however, the length scale $l_L$ is believed to be comparable to the transverse Eulerian integral (length) scale $l$, 

\[ \overline{u_2^2} l = \int_0^\infty u_2(x_2)u_2(0)dx_2 \]

• If $l$ and $l_L$ are of the same order of magnitude, then 

\[ \frac{1}{2} \frac{d}{dt} \left( \overline{x_2^2} \right) = \overline{x_2 u_2} = c_1 u'_2 l \]
The Gradient Transport Fallacy

- Let us demonstrate that

\[ \tau_{12} = c_1 \rho u'_2 l \frac{\partial U_1}{\partial x_2} = -\rho u_1 u_2 \]

is merely a dimensional necessity in a turbulent shear flow dominated by a single velocity scale \( u'_2 \) and a single length scale \( l \).

- Since \( c_{12} = \frac{u_1 u_2}{u'_1 u'_2} \) then \( \tau_{12} = -c_{12} \rho u'_1 u'_2 \)

- In all turbulent flows, \( u'_1 \) and \( u'_2 \) are of the same order of magnitude, then

\[ \tau_{12} = c_2 \rho \left( u'_2 \right)^2 \]

- In turbulent flow driven shear, the unknown coefficients \( c_{12} \) and \( c_2 \) are always of order one: \( u_1 \) and \( u_2 \) are well correlated in eddies that can absorb energy from the mean flow by vortex stretching.
The eddies involved in momentum transfer have characteristic vortices of order \( u'_2 / l \) which is maintained by their interaction with mean flow, then \[ \frac{u'_2}{l} = c_3 \frac{\partial U_1}{\partial x_2} \]

where \( c_3 \) is a non-dimensional coefficient of order one.

In effect, we are merely saying that the characteristic time of eddies, \( l / u'_2 \) and the mean flow \((\partial U_1 / \partial x_2)^{-1} \) should be of the same order, if no other characteristic times or lengths are present.

In particular, it is implied that \( l \) and \( L = (\partial U_1 / \partial x_2) / (\partial^2 U_1 / \partial x_2^2) \) are of the same order.

Therefore, the stress can be expressed as,

\[ \tau_{12} = c_2 \rho (u'_2)^2 = c_2 c_3 \rho u'_2 l \frac{\partial U_1}{\partial x_2} \]

The stress at \( x_2 = 0 \) can be related to the mean velocity gradient at \( x_2 = 0 \) because the correlation between \( u_1 \) and \( u_2 \) is good and because the time scale ratio is of order one.
Further Estimates

- If the correlation between $u_1$ and $u_2$ is good and if $u'_1$ and $u'_2$ are of the same order, we may write

$$\tau_{12} = c_4 \rho l^2 \left. \frac{\partial U_1}{\partial x_2} \right| \left. \frac{\partial U_1}{\partial x_2} \right|$$  \hspace{1cm} (Prandtl, 1959)

$c_4$ is a coefficient of order one; the modulus of $\partial U_1/\partial x_2$ is used to make $\tau_{12}$ switch signs with $\partial U_1/\partial x_2$.

- The ratio of the Reynolds stress to the viscous stress is,

$$\frac{\tau_{12}}{\mu \partial U_1 / \partial x_2} = \frac{\rho \nu_t \partial U_1 / \partial x_2}{\mu \partial U_1 / \partial x_2} = \frac{\nu_t}{\nu} = \frac{c_1 u'_2 l}{\nu} = c_1 \text{Re}$$

- In most flows, Re is very large, which implies that Reynolds stress is much larger than viscous stress.
Recapitulation

- In a shear flow with one characteristic velocity and one characteristic length, the time scale of the turbulence is proportional to the time scale of the mean flow.
- Under certain circumstances, \((l/u_2')\) may be as small as one-tenth of the reciprocal of \(\partial U_1/\partial x_2\), turbulence shear flow, however, can not be in a state of equilibrium as will try to adjust to its environment.
- In all turbulent shear flows \(\left| -u_1u_2 \right| \sim 0.4u_1'u_2'\); the value of 0.4 should be contrasted to the correlation coefficient for molecular motion, which was to be of order 10\(^{-6}\).
- A theory for Reynolds stress thus can not be patterned after the kinetic theory of gases; the mixing-length model must be rejected, even though a mixing length expression makes good dimensional sense in a situation where only one length scale and only one time scale are relevant.
Turbulent Heat Transfer

- Passive contaminants are transported by turbulent motions in much the same way as momentum.
- It is also assumed here that heat flux does not cause significant buoyancy effects.
Reynolds Analogy

- Vertical heat flux
  \[ H_2 = \rho C_p \frac{\partial u_2}{\partial t} = -\rho C_p \gamma_t \frac{\partial \Theta}{\partial x_2} \quad \text{and} \quad \tau_{12} = -\rho u_1 u_2 = \rho v_t \frac{\partial U_1}{\partial x_2} \]

- If \( \nu_t / \gamma_t \) is equal to one, then
  \[ \frac{H_2}{C_p \tau_{12}} = -\gamma_t \frac{\partial \Theta / \partial x_2}{\rho v_t \partial U_1 / \partial x_2} = -\frac{\partial \Theta}{\partial x_2} / \frac{\partial U_1}{\partial x_2} \]

- This is called Reynolds’s analogy.
- It is used to estimate the turbulent heat flux if the stress, the mean velocity and the temperature fields are known.
- The analogy avoid an explicit statement on the magnitudes of the eddy diffusivity for heat and momentum, so that it can be applied even if \( \nu_t \) and \( \gamma_t \) can not be determined.
The Mixing Length Model

- If the correlation between $u_2$ and $\theta$ is good and if $\theta'_2/l \sim \partial \Theta / \partial x_2$
- The heat flux can be expressed as,

$$H_2 = -\rho C_p c_5 u'_2 l \frac{\partial \Theta}{\partial x_2}$$

- The expression $\theta'_2/l \sim \partial \Theta / \partial x_2$ often is more reliable than its momentum counterpart $u'_2/l \sim \partial U_1 / \partial x_2$. Because the former merely expresses that turbulence mixes passive scalar contaminants over scales of order $l$ whereas the latter is valid only if the turbulent motion is maintained by a mean strain rate.
- Momentum is not a passive contaminant; “mixing” of mean momentum relates to the dynamics of turbulence, not only to its kinematics.
Turbulent Shear Flow Near a Rigid Wall

- Consider a pure shear flow in the vicinity of a rigid, but porous wall. (Fig. 2.8)
- If there is no mass transfer (blowing or suction) through the wall, there is only one velocity scale and mixing length models may be used.
- However, if the mass transfer velocity is different from zero, there are two velocity scales.
- Take the mean flow to be steady and homogeneous in $x_1, x_3$ plane and $U_3=0$ and $\partial P/\partial x_i=0$, $i=1, 2, 3$.
- The equations of motion are,

\[
\frac{\partial U_2}{\partial x_2} = 0, \quad U_2 \frac{\partial U_1}{\partial x_2} = \frac{1}{\rho} \frac{\partial}{\partial x_2} T_{12}
\]
• From the continuity equation, \( U_2 = v_m \), thus the momentum equation can be integrated to yield,

\[
\rho v_m U_1 = T_{12} - T_{12}(0)
\]

where \( U_1(0) = 0 \).

• Define a friction velocity, \( u_* \) by \( T_{12}(0) = \rho u_*^2 \). If the analysis is restricted to values of \( x_2 \) where \( x_2 U_1/\nu >> 1 \), the viscous contribution to the total shear stress \( T_{12} \) should be negligible. Thus,

\[
\nu_m U_1 = -u_1 u_2 - u_*^2
\]
A Flow with Constant Stress

- If $v_m=0$, the Reynolds stress $-u_1'u_2$ is equal to $u_*^2$ at all values of $x_2$ for which viscous effects are negligible.
- A flow of this kind is called a constant-stress layer; it also occurs close to the wall in most turbulent boundary layer.
- Assuming that $u_1$ and $u_2$ are well correlated, $u'_2$ must be independent of $x_2$ and proportional to $u_*$.
- The scale relation between the vorticity of the turbulence and of the mean flow becomes,

$$u_*/l = \alpha_1 \partial U_1 / \partial x_2$$

where $\alpha_1$ is a coefficient of order one.
- The rigid wall constrains the turbulence motion that transport of momentum downward from some level $x_2$ is restricted to distances smaller than $x_2$ itself.
• If no length scales are imposed on this flow, the only dimensionally correct choice for \( l \) is \( l = \alpha_1 x_2 \). Thus,

\[
\frac{\partial U_1}{\partial x_2} = \frac{u_*}{\kappa x_2}, \quad \frac{U_1}{u_*} = \frac{l}{\kappa} \ln x_2 + \text{constant}
\]

where \( \kappa \) is known as the constant of von Karman which is approximately equal to 0.4. \textbf{(Universal wall law)}

• In this flow without mass transfer through the surfaces, mixing length models can be used because there is only one length scale \((x_2)\) and one velocity scale \((u_*)\).

\[
-u_1 u_2 = \kappa u_* x_2 \frac{\partial U_1}{\partial x_2}
\]

• Because

\[
u_*^2 = -u_1 u_2 = \kappa u_* x_2 \frac{\partial U_1}{\partial x_2}, \quad \frac{u_*}{\kappa x_2} = \frac{\partial U_1}{\partial x_2}
\]

Prandtl’s version of the mixing length formula can be applied with equal success.
Nonzero Mass Transfer

- If \( v_m \neq 0 \), the problem has two characteristic velocities \( u_* \) and \( v_m \).
- The length scale, however, remains proportional to \( x_2 \).
- Assuming that \( \partial U_1/\partial x_2 \) should be proportional to \( w/x_2 \), where \( w \) is an undetermined velocity scale that depends on \( u_* \) and \( v_m \).
  \[
  \frac{\partial U_1}{\partial x_2} = \frac{w}{x_2}
  \]
- Integration yields, \( U_1/w = \ln x_2 + \text{constant} \)
- Because \( w \) is unknown, it has to be determined experimentally.
- In this flow, \( u_* \) and \( v_m \) are the only velocity scales.
  \[
  \frac{w}{u_*} = f(\frac{v_m}{u_*}) \quad \text{(Fig. 2.9)}
  \]
• In the case of blowing ($\nu_m > 0$), the Reynolds stress is larger than $u_*^2$; this results in an increase of $w/u_*$.  
  \[ \nu_m U_1 = -u_1 u_2 - u_*^2 \]
• If $\nu_m >> u_*$, the friction velocity becomes relatively unimportant, so that $w$ should be proportional to $\nu_m$.
• In the case of suction ($\nu_m < 0$), the Reynolds stress is smaller than $u_*^2$, so that $w/u_*$ decreases.
• If the suction rate is large, the Reynolds stress becomes so small that turbulence cannot be maintained, this causes reverse transition from turbulent to laminar flow.
The Mixing Length Approach

- From, \( v_m U_1 = -u_1 u_2 - u_*^2 \), \( U_1/w = \ln x_2 + \text{constant} \)
  
  then \( -u_1 u_2 = u_*^2 + v_m w (\ln x_2 + c) \)

- If we insist on using mixing length model and if we continue to use \( w \) as a characteristic velocity, then

  \[
  -u_1 u_2 = \alpha_3 w x_2 \frac{\partial U_1}{\partial x_2}
  \]

  where \( \alpha_3 \) is an unknown coefficient. Since \( \frac{\partial U_1}{\partial x_2} = w / x_2 \)
  then \( -u_1 u_2 = \alpha_3 w^2 \).

- A stress that is independent of \( x_2 \) is clearly not a correct solution; because \( U_1 \) depends on \( x_2 \), stress should also depend on \( x_2 \).

\[
 v_m U_1 = -u_1 u_2 - u_*^2
\]
The Limitation of Mixing Length Theory

- Mixing length models are incapable of describing turbulent flows containing more than one characteristic velocity with any degree of consistency.
- None of the versions gives a clear picture of the roles of the two velocity scale; the effects of $\nu_m/u_*$ on the integration constants remain unsolved.
- Mixing length expression can be understood as the combination of a statement about the stress $-u_1 u_2 \sim w^2$ and a statement about the mean velocity gradient $\partial U_1 / \partial x_2 \sim w / x_2$.
- These statements do not give rise to inconsistencies if there is only one characteristic velocity.
The Dynamic of Turbulence

- Two major questions arise,
  First, how is the kinetic energy of the turbulence maintained?
  Second, why are vorticity and vortex stretching so important to the study of turbulence?
- To help answer these questions, we shall proceed as follows. We first derive equations for the kinetic energy of the mean flow and that of the turbulence.
- We shall see that turbulence extracts energy from the mean flow at large eddies and that this gain is approximately balanced by viscous dissipation at very small scales.
In order to gain an appreciation of the role of the vorticity fluctuations, we first analyze how they are involved in the generation of Reynolds stresses. It turns out to be convenient to associate the Reynolds shear stress with transport and stretching of vorticity. With the understanding obtained that way, the vorticity equations can be studied.

Energy is transferred to small scales by vortex stretching and that the dissipation rate is proportional to the mean square vorticity fluctuation if the Reynolds number is large.
Kinetic Energy of the Mean Flow

- Equations of motion for steady mean flow in an incompressible fluid are,

\[ U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{T_{ij}}{\rho} \right) \]

where

\[ T_{ij} = -P \delta_{ij} + 2 \mu S_{ij} - \rho u_i u_j, \quad S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \]

- The equation governing the dynamics of the mean flow energy, \( K = U_i U_i / 2 \), is obtained by multiplying the equations of motion by \( U_i \), then

\[ \rho U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_i \right) = \frac{\partial}{\partial x_j} \left( T_{ij} U_i \right) - T_{ij} \frac{\partial U_i}{\partial x_j} \]
• The strain rate can be decomposed into two parts,

\[
\frac{\partial U_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) = S_{ij} + \Omega_{ij}
\]

• Because \( T_{ij} \) is a symmetric tensor,

\[
T_{ij} \cdot \Omega_{ij} = 0; \quad T_{ij} \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right) = 0
\]

• Thus the mean kinetic energy equation becomes,

\[
\rho U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_i \right) = \frac{\partial}{\partial x_j} \left( T_{ij} U_i \right) - T_{ij} S_{ij}
\]
The first term on the right hand side of the mean kinetic energy equation represents transport of mean energy by the stress $T_{ij}$. This term integrates to zero if the integration refers to a control volume on whose surface either $T_{ij}$ or $U_i$ vanishes.

According to the divergence theorem,

$$
\int_V \frac{\partial}{\partial x_j} \left( T_{ij} U_i \right) dV = \int_s n_j T_{ij} U_i ds
$$

where $n_j$ is a unit vector normal to the surface element $ds$.

If the work performed by the stress on the surface $s$ of the control volume $V$ is zero, only the volume integral of $T_{ij} S_{ij}$ can change the total amount of kinetic energy.

The term $T_{ij} S_{ij}$ is called deformation work; by virtue of conservation of energy, it represents kinetic energy of the mean flow that is lost to or retrieved from the agency that generates the stress.
Pure Shear Flow

- All variables depend on $x_2$ only and in which the only nonzero component of $U_i$ is $U_1$.
- For this turbulent Couette flow, the energy equation becomes,

$$
0 = \frac{\partial}{\partial x_2} (T_{12} U_1) - T_{12} \frac{\partial U_1}{\partial x_2}
$$

which illustrates that the rate of work done by the stress per unit volume is equal to the first term in the above equation and the equation also implies that $T_{ij}$ is constant.

- A constant stress field does not accelerate a flow; the tendency to change $U_i U_i/2$ by $\partial (T_{12} U_1)/\partial x_2$ is balanced by the deformation work $T_{12} \partial U_1/\partial x_2$. 

We expect that deformation work generally will be an input term for the energy of the agency that generates the stress and that the kinetic energy $K$ will decrease because of the deformation work unless this loss is balanced by a net input of energy.

The deformation work is caused by the stresses that contribute to $T_{ij}$.

$$T_{ij}S_{ij} = 2\mu S_{ij}S_{ij} - \rho u_iu_j S_{ij}$$

The contribution of the pressure to deformation work in an incompressible fluid is zero.

$$-P\delta_{ij}S_{ij} = -PS_{ii} = -P\frac{\partial U_i}{\partial x_i} = 0$$

The contribution of viscous stresses to the deformation work is always negative; consequently, viscous deformation work always represents a loss of kinetic energy. This term, $2\mu S_{ij}S_{ij}$ is called viscous dissipation which is related to the strain rate, not to the vorticity.
The contribution of Reynolds stress to the deformation work is also dissipative in most flow: negative values of $u_i u_j$ tend to occur in situations with positive $S_{ij}$.

Positive values of $u_i u_j S_{ij}$ can occur in unusual situations; even then the region in which $u_i u_j S_{ij} > 0$ is a small fraction of the entire flow.

Since turbulent stresses perform the deformation work, the kinetic energy of the turbulence benefits from this work; for this reason, $-u_i u_j S_{ij}$ is known as turbulent energy production.
The Effect of Viscosity

- The energy equation of the mean flow is,

\[ U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} U_i U_i \right) = \frac{\partial}{\partial x_j} \left( -\frac{1}{\rho} P U_i + 2\nu S_{ij} U_i - \bar{u}_i \bar{u}_j U_i \right) - 2\nu S_{ij} S_{ij} + \bar{u}_i \bar{u}_j S_{ij} \]

- The first three terms on the right hand side are called pressure work, transport of mean flow energy by viscous stresses, and transport of mean flow energy by Reynolds stress, respectively.

- The word “transport” refers to the integral property within the divergence; if \( U_i T_{ij} \) is zero on the surface of a control volume, the first three terms can only redistribute energy inside the control volume.

- In most flow, the viscous terms \( (2\nu S_{ij} U_i, 2\nu S_{ij} S_{ij}) \) in the mean kinetic energy equation are negligible.
• If the turbulence is characterized by $u$ and $l$ and if no other characteristic scales are present,

$$\frac{\partial U_i}{\partial x_j} \sim \frac{u}{l}$$

where $l$ is an integral scale and the stress estimate

$$-u_iu_j \sim u^2 = \frac{1}{3} u_iu_i$$

• Turbulence production is estimated as, $(S_{ij} \sim u/l)$

$$-u_iu_jS_{ij} = c_1ulS_{ij}S_{ij} \leftrightarrow 2\nu S_{ij}S_{ij}$$

• Energy transport of turbulent motion is estimated as,

$$-u_iu_jU_i = c_2ulU_iS_{ij} \leftrightarrow 2\nu U_iS_{ij}$$

• In most simple shear flows, the undetermined coefficients $c_1$ and $c_2$ are of order one.
- Turbulence terms are $ul/\nu$ times as large as the viscous terms.
- This Reynolds number tends to be very large (except in situation very close to the smooth surface), so that the viscous terms can ordinarily be neglected.
- This conclusion again illustrates that the gross feature of turbulent flows tends to be virtually independent of viscosity. Viscosity makes itself felt only indirectly.
Turbulent Boundary Layer Over a Flat Plate

- The mean kinetic energy equation becomes,

\[
\frac{\partial U K}{\partial x} + \frac{\partial V K}{\partial y} = -\frac{1}{\rho} \frac{\partial (UP)}{\partial x} + \frac{\partial}{\partial y}\left(\nu U \frac{\partial U}{\partial y}\right) - \nu \left(\frac{\partial U}{\partial y}\right)^2 - \frac{\partial}{\partial y}(U \overline{uv}) + \overline{uv} \frac{\partial U}{\partial y}
\]

- A: The rate of increase of mean kinetic energy $K$
- B: transport of mean flow energy by viscous stresses
- C: viscous dissipation
- D: transport of mean flow energy by Reynolds stress
- E: turbulent energy production
Laminar Flow Over Flat Plate with Negligible Pressure Gradient

- Terms D, E vanishes.
- Near an impermeable wall the convection terms are negligible and hence a balance must exist between stresses; net work is being done on a fluid particle at the same rate as it is losing kinetic energy due to the dissipative action of viscosity.
- The kinetic energy of the particle remains effectively constant as it flows along.
- The integration of term (B) across the boundary layer being zero indicates that this term makes no contribution to the overall rate of change of kinetic energy. It acts to redistribute energy from regions of higher to regions of low kinetic energy.
- In this region \((10\%y/\delta)\), the pressure gradient and convective transport are negligible.

- Immediately adjacent to the surface, turbulent stresses are negligible and there is a balance between term (B) and (C).

- Somewhat further from the surface all four terms (B), (C), (D) and (E) are important.

- Still further away from the wall the viscous terms are swamped by turbulent interactions; then there is a balance between terms (D) and (E).

- The term (D) is directly analogous to term (B); it makes no contribution to the overall rate of change of mean kinetic energy, since \(\overline{uw}\) vanishes at a free boundary and also at the wall. It is a diffusor.

- Term (E) represents the rate at which kinetic energy is transferred from the mean flow to the turbulent motion.
- Where (E) is maximum?

\[ \frac{\partial}{\partial y} \left( \bar{u} \frac{\partial U}{\partial y} \right) = 0 \]

or,

\[ \frac{\partial \bar{u} \bar{v}}{\partial y} \frac{\partial U}{\partial y} + \bar{u} \frac{\partial^2 U}{\partial y^2} = 0 \]

- Now if convective terms are negligible, the total stress \( \tau \) at any point in the boundary layer is constant and equal to wall stress \( \tau_w \), i.e.

\[ \frac{\tau_w}{\rho} = \nu \frac{\partial U}{\partial y} - \bar{u} \bar{v} \quad \text{or}, \quad \bar{u} \bar{v} = \frac{\tau_w}{\rho} - \nu \frac{\partial U}{\partial y} \]

- Insert the relation into the term \( \partial \bar{u} \bar{v} / \partial y \),

\[ - \frac{\partial U}{\partial y} \left[ \frac{\partial}{\partial y} \left( \frac{\tau_w}{\rho} - \nu \frac{\partial U}{\partial y} \right) \right] = -\bar{u} \bar{v} \frac{\partial^2 U}{\partial y^2} \]

\[ \nu \frac{\partial U}{\partial u} \frac{\partial^2 U}{\partial u^2} = -\bar{u} \bar{v} \frac{\partial^2 U}{\partial y^2} \]

Therefore,

\[ \nu \frac{\partial U}{\partial y} = -\bar{u} \bar{v} \]

- It indicates that the rate at which the mean flow loses kinetic energy to turbulence is a maximum where viscous and turbulent shear stresses are equal.

- It is apparent that (C) and (E) are equal at this region and the magnitude of (D) exceeds that of (E).

- Beyond the region where the viscous terms are significant, the gain in energy by the diffusive action of the turbulent motion just balances the loss of mean-flow energy to turbulence.

- Measurements indicate that the loss of kinetic energy to turbulence is greatest at \( y^+ \approx 15 \).
Mean Energy Balance in the Outer Region of Flat Plate Boundary Layer

- The figure shows the results of a mean energy balance for the outer 80% of the boundary layer.
- Viscous terms (B) and (C) are negligible and are omitted.
- To the left of the diagram, there is a balance between (D) and (E).
- The loss of mean kinetic energy to turbulence falls monotonically to zero and is negligible small over the 40% of the layer.
- The turbulent diffusion term changes sign about $y/\delta \sim 0.3$ and is negative - representing a net energy withdraw over the whole outer region.
- Thus a small fluid package in the outer region losses energy by diffusion to fluid nearer the wall (with less kinetic energy) at a faster rate than it is gaining energy from the faster moving fluid adjacent to it on the side remote from the wall.
- Consequently, the fluid element will lose mean kinetic energy as it moves along, as shown by (A).
Mean Energy Balance in a Plane Turbulent Jet

- The plane jet is an example of free shear flows, i.e. flows where no solid surface exerts significant effect on the flow development.

- Viscous terms make negligible contribution to the mean energy transfer processes, so there is a balance between (A), (D) and (E).

- The greatest rate of mean energy loss to turbulence occurs midway across the jet where the mean velocity gradient is largest.

- There is a strong diffusive transfer of energy from the inner region of the jet outwards.

- Most of the jet fluid elements are losing kinetic energy as the flow develops downstream.

- Over the outer quarter of the jet, newly entrained fluid accelerates (for a while), as it flows downstream, the energy being supplied by a diffusive transport from the higher velocity fluid element near the axis.
The equation governing the mean kinetic energy $k = \overline{u_i u_i}/2$ of the turbulence velocity fluctuation is obtained by multiplying the equation for $u_i$ by $u_i$, then taking the time average of all terms. The final equation, turbulent energy budget, reads:

$$ U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \overline{u_i u_i} \right) = - \frac{\partial}{\partial x_j} \left( \frac{1}{\rho} \overline{u_i p} + \frac{1}{2} \overline{u_i u_j} \overline{u_j} - 2\nu \overline{u_i} \overline{s_{ij}} \right) - \overline{u_i u_j s_{ij}} - 2\nu \overline{s_i s_j} $$

where

$$ s_{ij} = \frac{1}{2} ( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} ) $$

is the fluctuating rate of strain.

- The rate of change of $\overline{u_i u_i}/2$ is thus due to the pressure gradient work, transport by turbulent velocity fluctuations, transport by viscous stresses, and two kinds of deformation work.

- The transport terms are divergence of energy flux; if the energy flux out of or into a closed control volume is zero, these terms merely redistribute energy from one point in the flow to another.

- The deformation-work terms are more important.

- The turbulence production $-\overline{u_i u_j} s_{ij}$ occurs in the turbulence and mean kinetic energy equations with opposite sign.

- This term apparently serves to exchange kinetic energy between the mean flow and the turbulence; normally the energy exchange involves loss to the mean flow and a profit to the turbulence.

- $2\nu \overline{s_i s_j}$ is the rate at which viscous stresses perform deformation work against the fluctuating strain rate. This always is a drain of energy, since the term is quadratic in $s_{ij}$ and is called viscous dissipation which is essential to the dynamics of turbulence and can not ordinarily be neglected.
Production Equals dissipation

- In a steady, homogeneous, pure shear flow (in which all averaged quantities except $U_i$ are independent of position and in which $S_{ij}$ is a constant), the turbulence kinetic energy equation reduces to,

$$-\overline{u_i u_j S_{ij}} = 2\nu \overline{s_{ij} s_{ij}}$$

This equation states that in this flow the rate of production of turbulent energy by Reynolds stresses equals the rate of viscous dissipation.

- In most shear flows production and dissipation do not balance, though they are nearly always of the same order of magnitude.

- Customarily, production and dissipation are often written in symbolic forms,

$$P \equiv -\overline{u_i u_j S_{ij}}, \quad \epsilon \equiv 2\nu \overline{s_{ij} s_{ij}}$$

Therefore $P = \epsilon$. 
For a shear-generated turbulence with one length scale and one velocity scale, we may employ

\[ S_{ij} \sim \frac{u}{\ell} \quad -\bar{u}_i\bar{u}_j \sim u^2 \]

Thus the energy budget \( P = \varepsilon \) becomes,

\[ c_1 \ell S_{ij} S_{ij} = 2\nu s_{ij}s_{ij} \]

\[ c_1 \frac{\ell}{\nu} S_{ij} S_{ij} = 2\bar{s}_{ij}s_{ij} \]

Since the Reynolds number \( \frac{ul}{\nu} \) is generally very large, we conclude that

\[ \frac{s_{ij}s_{ij}}{S_{ij}S_{ij}} \gg \frac{s_{ij}s_{ij}}{S_{ij}S_{ij}} \]

- The fluctuating strain rate \( s_{ij} \) is thus very much larger than the mean rate of strain \( S_{ij} \) when the Reynolds number is large.

- Since the strain rates have the dimensions of \( \text{sec}^{-1} \), this implies that the eddies contributing most to the dissipation of energy have very small convective time scales compared to the time scale of the flow.

- This suggests that \( S_{ij} \) and \( s_{ij} \) do not interact strongly if the Reynolds number is large, because they are not tuned to the same frequency band.

- Therefore, the small scale structure of turbulence tends to be independent of any orientation effects introduced by the mean shear.

- Therefore all averages relating to the small eddies do not change under rotations of the co-ordinate system; the small structure is called isotropic.

- Isotropy at small scales is called local isotropy.

\textbf{Fig. 3.2}
Taylor Microscale

- Any length scale involved in estimates of $s_{ij}$ must be very much smaller than $\ell$ if a balance between production and dissipation is to be obtained.

- The situation is very similar to the one in laminar boundary-layer theory.

\[
\frac{U^2}{L} \sim \nu \frac{U}{\delta^2}
\]

\[
\frac{\delta}{L} \sim \left(\frac{\nu}{UL}\right)^{1/2} = R^{-1/2}
\]

- The dissipation of energy is proportional to $s_{ij} s_{ij}$; this consists of several terms like $(\partial u_i/\partial x_j)^2$, most of which can not be measured conveniently.

- However, the small-scale structure of turbulence tends to be isotropic.

- In isotropic turbulence, the dissipation rate is equal to,

\[
\epsilon = 2\nu s_{ij} s_{ij} = 15\nu (\frac{\partial u_1}{\partial x_1})^2
\]

In many flows, $(\partial u_1/\partial x_1)^2$ can be measured relatively easily.
Let us define a new length scale, $\lambda$,

$$\left(\frac{\partial \bar{u}_1}{\partial x_1}\right)^2 = \frac{\bar{u}_1^2}{\lambda^2} = \frac{u^2}{\lambda^2}$$

where in isotropic turbulence $\bar{u}_1^2 = \bar{u}_2^2 = \bar{u}_3^2 = u^2 = \bar{u}_i \bar{u}_i / 3$

The length scale $\lambda$ is called the Taylor microscale in honour of G. I. Taylor.

The small scale structure of turbulence at large Reynolds number is always approximately isotropic. Then,

$$\epsilon = 15\nu \frac{u^2}{\lambda^2}$$

A relation between $\lambda$ and $\ell$ can be obtained from the simplified energy budget and $S_{ij} \sim u / \ell$, $-\bar{u}_i \bar{u}_j \sim u^2$. Then,

$$A \frac{u^3}{\ell} = 15\nu \frac{u^2}{\lambda^2}$$

$$\frac{\lambda}{\ell} = \left(\frac{15}{A}\right)^{1/2} \left(\frac{u_\ell}{\nu}\right)^{-1/2} = \left(\frac{15}{A}\right)^{1/2} R_t^{-1/2}$$

where $A$ is an undetermined constant, which is presumably of order one.

Because in all turbulent flow $R_t >> 1$, the Taylor microscale $\lambda$ is always much smaller than the integral scale.

This indicates that dissipation of energy is due to the small eddies of turbulence.
Scale Relations

- The Taylor microscale is not the smallest length scale occurring in turbulence.
- The smallest scale is the Kolmogorov microscale $\eta$, $\eta = (\nu^3/\varepsilon)^{1/4}$.
- The strain-rate fluctuations $s_{ij}$ have the dimension of a frequency ($sec^{-1}$); the definition of $\varepsilon$ thus defines a time scale associated with the dissipative structure of turbulence.
- Calling this time scale $\tau$, then
  \[ \tau = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \]
- The dimensions of $s_{ij}$ are such that the length scale $\lambda$ was found by taking $u$ as a velocity scale.
  The Taylor microscale should thus be used only in $\varepsilon = 15\nu u^2/\lambda^2$.
  \[ \frac{u}{\lambda} = 0.26\tau^{-1} = 0.26\left(\frac{\varepsilon}{\nu}\right)^{1/2} \]
- The Taylor microscale is thus not a characteristic length of the strain-rate field and does not represent any group of eddy size in which dissipative effects are strong.
- It is not a dissipation scale, because it is defined with the assistance of a velocity scale which is not relevant for the dissipative eddies.
- $\lambda$ is used frequently because the estimate $s_{ij} \sim u/\lambda$ is often convenient.
- Expressions relating $\ell$, $\lambda$ and $\eta$ are,

$$A \frac{u^3}{\ell} = 15\nu \frac{u^2}{\lambda^2}; \quad \lambda^2 = \frac{15\nu \ell}{Au}$$

$$\frac{\lambda}{\ell} = \left(\frac{15}{A}\right)^{1/2} \left(\frac{u\ell}{\nu}\right)^{-1/2} = \left(\frac{15}{A}\right)^{1/2} R^{-1/2}_\ell = \frac{15}{A} \left(\frac{u\lambda}{\nu}\right)^{-1} = \frac{15}{A} R^{-1}_\lambda$$

$$\eta^4 = \frac{\nu^3}{\epsilon}; \quad \epsilon = \frac{\nu^3}{\eta^4} = 15\nu \frac{u^2}{\lambda^2}$$

$$\frac{\lambda}{\eta} = 15^{1/4} \left(\frac{u\lambda}{\nu}\right)^{1/2} = 15^{1/4} R^{1/2}_\lambda$$

$$\frac{\lambda}{\eta} = \left[\frac{15\nu u^2}{\nu^3} \frac{15\nu \ell}{Au}\right]^{1/4} = \left(\frac{225}{A}\right)^{1/4} \left(\frac{u\ell}{\nu}\right)^{1/4} = \left(\frac{225}{A}\right)^{1/4} R^{1/4}_\ell$$

- The parameter $R_\lambda$ is the microscale Reynolds number

$$R_\lambda = \frac{u\lambda}{\nu} = \frac{\ell/u}{\lambda/u} = \frac{\lambda^2/\nu}{\lambda/\nu}$$

which may be interpreted as the ratio of large eddy time scale $\ell/u$ (which is proportional to $\lambda^2/\nu$) and the time scale of the strain-rate fluctuation.
Spectral Energy Transfer

- The energy exchange between the mean flow and the turbulence is governed by the dynamics of the large eddies.

- Large eddies contribute most to the turbulence production $P$, because $P$ increases with eddy size.

- Turbulence energy extracted from the mean flow occurs mainly at scales comparable to the integral scale $\ell$.

- The viscous dissipation of turbulent energy occurs mainly at scales comparable to the Kolmogorov microscale $\eta$.

- This implies that internal dynamics of turbulence must transfer energy from large scales to small scales.

- Available experimental evidence suggests that this spectral energy transfer precedes at a rate dictated by the energy of the large eddies (which is of order $u^2$) and their time scale ($u/\ell$).

- Thus the dissipation rate may always be estimated as,

$$\epsilon = A \frac{u^3}{\ell}$$

provided there exists only one characteristic length $\ell$. 
- This estimate is independent of the presence of production; and is thus a valid statement about the dissipation rate even if production and dissipation do not balance.

- The approximate balance between $P$ and $\epsilon$ which occurs in many turbulent flows may be written as,

$$-\overline{u_i u_j} S_{ij} \sim A \frac{u^3}{\ell}$$

- With $\epsilon$ of order $u^3/\ell$ because spectral energy transfer is of that order and with $S_{ij}$ of order $u/\ell$ because the vorticity of the large scale is maintained by the vorticity and the strain rate of the mean flow, we conclude that $-\overline{u_i u_j}$ has to be of order $u^2$ if a balance between $P$ and $\epsilon$, however approximate, is to be obtained.

- This also indicates that good correlation between $u_i$ and $u_j$ can be obtained only if $S_{ij}$ and $u/\ell$ occur in the same range of frequencies.
Further Estimates

- The orders of magnitude of the other terms of the turbulence kinetic energy budget need to be estimated.
  
  - We shall use
  
  \[ s_{ij} \sim \frac{u}{\lambda}; \quad \frac{\lambda}{\ell} \sim R_t^{-1/2} \]

- The pressure work term is estimated as,
  
  \[ -\frac{\partial}{\partial x_j} \left( \frac{1}{\rho} \frac{u_i p}{u_j} \right) \sim \frac{u^3}{\ell} \]

  because the pressure fluctuation \( p \) should be of order \( \rho u^2 \).

- Mean transport of turbulent energy by turbulent motion is estimated as,
  
  \[ -\frac{\partial}{\partial x_j} \left( \frac{1}{2} \frac{u_i u_i u_j}{u_j} \right) \sim \frac{u^3}{\ell} \]

- It is tempting to estimate transport of viscous stresses as\(^2\),
  
  \[ 2\nu \frac{\partial}{\partial x_j} (u_i s_{ij}) \sim \frac{\nu u^2}{\ell} \frac{u^3}{\lambda} \sim \frac{u^3}{\ell} R_t^{-1/2} \]

  This estimate, however, is too large because it assumes that \( u_i \) and \( s_{ij} \) are well correlated.

  This is not likely because the scales of the eddies contributing most to \( s_{ij} \) is much smaller than the time scale of the eddies contributing most to \( u_i \).
- The problem can be easily resolved,

\[ 2\nu \frac{\partial}{\partial x_j} (u_i s_{ij}) = \nu \frac{\partial}{\partial x_j} (u_i \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_i}) = \nu \frac{\partial^2}{\partial x_j \partial x_i} \left( \frac{1}{2} u_i u_i \right) + \nu \frac{\partial^2}{\partial x_j \partial x_i} (u_i u_j) \sim \nu \frac{u^2}{\ell^2} \sim \frac{u^3}{\ell} R_t^{-1} \]

Thus the correlation coefficient between \( u_i \) and \( s_{ij} \) must be of order \( R_t^{-1/2} \).

- The time scale of the large eddies is \( \ell/u \) and the time scale of the dissipative eddies is of order \( \lambda/u \).

- The ratio of these time scales is \( \lambda/\ell \) which is of order \( R_t^{-1/2} \).

- The correlation coefficient thus scales with the ratio of the time scales involved.

- \( u_i \) and \( s_{ij} \) can not interact strongly at large Reynolds number because they are not tuned to the same frequency range.

- The dimensional estimate indicates that only the viscous transport of turbulent energy can be neglected if the Reynolds number is large.
Wind Tunnel Turbulence (Fig. 3.3)

- Wind tunnel turbulence is commonly generated by a grid or screen in a uniform flow without shear.
- If $S_{ij}$ is zero, there is no turbulence production; the turbulence should decay through viscous dissipation.
- If $U_1$ is the only nonzero component of the mean quantity, the energy budget reads,

$$U_1 \frac{\partial}{\partial x_1} \left( \frac{1}{2} u_i u_i \right) = - \frac{\partial}{\partial x_1} \left( \frac{1}{\rho} u_1 p + \frac{1}{2} u_i u_i u_1 \right) - \epsilon$$

assuming that $R_e$ is large.
- The orders of magnitude of the various terms maybe estimated as follows,

$$U_1 \frac{\partial}{\partial x_1} \left( \frac{1}{2} u_i u_i \right) = \mathcal{O} \left( \frac{U_1}{x_1} u^2 \right)$$

$$- \frac{\partial}{\partial x_1} \left( \frac{1}{\rho} u_1 p + \frac{1}{2} u_i u_i u_1 \right) = \mathcal{O} \left( \frac{u^3}{x_1} \right)$$

$$\epsilon = \mathcal{O} \left( \frac{u^3}{L} \right)$$

The downstream distance $x_1$ is the appropriate length scale in the estimate of the downstream decay of $\partial / \partial x_1$ which can scale only with $x_1$.
- If $u^2 \sim x_1^\alpha$ then,

$$\frac{\partial u^2}{\partial x_1} \sim \frac{\alpha u^2}{x_1} ; \quad \frac{\partial}{\partial x_1} \sim x_1^{-1}$$

- In grid turbulence, the velocity fluctuations are small $u << U$.
- Therefore, the turbulent transport terms should be neglected, so that the energy equation reduces to,

$$U_1 \frac{\partial}{\partial x_1} \left( \frac{1}{2} u_i u_i \right) = - \epsilon$$
- The dimensional estimates suggest that,
  \[
  \frac{U_1}{x_1} = \mathcal{C} \frac{u}{\ell}
  \]
  which states that the time scale of the flow is of the same order of magnitude as the time scale of the turbulence.

- To determine how $\ell$ and $u$ change downstream, another relation is needed to solve this problem.

- The time scale of energy transfer from the large eddies to the small eddies is $\tau \sim \ell/u$.

- The time scale associated with the decay of the large eddies themselves is $T \sim \ell^2/\nu$.

- The ratio of these time scales is,
  \[
  \frac{T}{\tau} \sim \frac{\ell u}{\nu}
  \]
  which suggests that at large values of $R_\ell = \ell u/\nu$ the large eddies are affected very little by direct dissipation.

- We now assume that these time scales and $x_1/U_1$ are the only independent variables of the problem.

- A relation between the independent and dependent variables in no-dimensional form may be written as,
  \[
  \frac{u \ell}{\nu} = f\left( \frac{x_1}{U_1 \tau}, \frac{\tau}{T} \right) = g\left( \frac{x_1}{U_1 \tau} \right) = g\left( \frac{x_1 u}{U_1 \ell} \right)
  \]

- The only way in which $u \ell$ can be a function of $u/\ell$ is by replacing that $g$ be a constant.$^3$

- Hence, wind tunnel turbulence in its initial period of decay (where $R_\ell \gg 1$), should have approximately constant Reynolds number.
- Since $R_\ell$ should be independent of $x_1$, we find,

$$\ell = C \frac{x_1}{U_1}$$

$$u^2 = \frac{1}{3} \overline{u_i u_i} = u \frac{U_1}{x_1} \frac{\ell}{C} = C \frac{\nu U_1 R_\ell}{x_1}$$

$$\ell = CC^{1/2}(R_\ell \nu)^{1/2} \left( \frac{U_1}{x_1} \right)^{1/2} \frac{x_1}{U_1} = CC^{1/2} \frac{x_1}{U_1} (R_\ell \nu)^{1/2}$$

- Therefore,

$$u^2 \sim \frac{1}{x_1}, \quad \ell \sim x_1^{1/2}$$

- Since,

$$\frac{\lambda}{\ell} \sim R_\ell^{-1/2}, \quad \frac{\lambda}{\eta} \sim R_\ell^{1/4}$$

Because $R_\ell$ is a constant, the ratios of $\ell/\lambda$ and $\ell/\eta$ are constant. Hence $\lambda$ and $\eta$ are also proportional to $x_1^{1/2}$.

- We conclude that turbulent energy decays as $x_1^{-1}$, while all length scales grow as $x_1^{1/2}$.

- Experimental evidence indicates that the predicted exponents are within 30% of the observed values.

- At large distances from the grid, the turbulence decays much faster than indicated in the preceding analysis.
2-D Thin Shear Layer at High Re

- The equation may be simplified to the following,

\[
U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} = -\overline{uw} \frac{\partial U}{\partial y} - \frac{\partial}{\partial y} (\overline{vk'} + \overline{vp}) - \nu \frac{\partial^2 u_i}{\partial x_j^2}
\]

\(A - convection \quad B - production \quad C - diffusion \quad D - dissipation\)

where \(k'\) denotes the instantaneous, i.e. the unaveraged kinetic energy \(u_i^2/2\).
Energy Budget in a Wall Boundary Layer in Zero Pressure Gradient

- Over a considerable portion of the boundary layer, the production and dissipation rates of turbulence energy, terms (B) and (D), are much larger than the other contribution; they are roughly of equal magnitude and of opposite sign.

- This implies that the majority of turbulence energy produced is dissipated in the same region of the boundary layer, a situation that is referred as local equilibrium.

- Over a region extending from about 0.2δ to 0.5δ the measurements indicate that fluid elements suffer a net loss of energy by diffusive transfer.

- In the outer region of the boundary layer, there is a gain of turbulent energy by diffusion which makes up for the small excess of dissipation over production and leads to a slow rate of increase of kinetic energy of newly entrained fluid.
The mean and turbulent energy flow process in a zero-pressure-gradient boundary layer may thus be summarised as follows,

* The Reynolds stress in the outer region promotes a transfer of mean kinetic energy towards the slower moving wall layer, $-\partial (\overline{uu}U)/\partial y$.
* In the inner region the energy in flow is partly dissipated directly by viscous action, $-\nu (\partial U/\partial y)^2$, and partly converted to turbulent energy.
* The greatest proportion of the turbulent energy created is directly dissipated into heat, $\nu \overline{s_{ij} s_{ij}}$.
* There is, however, a small excess of energy generation over dissipation and through diffusive transport of this surplus energy to the outer regions of the layer; newly entrained fluid slowly acquires turbulent energy.
Energy Budget of a Plane Jet

- In the plane jet, the dissipation term is appreciably more important and correspondingly the diffusion term less so.

- The left hand side of graph is the plane of symmetry and we find substantial rates of energy dissipation here.

- The energy lost is replenished to some extent by a diffusion energy, but there is a general decay of the energy level with passage downstream; the only exception is near the outer edge where newly entrained fluid increases its energy.
The Equations Governing the Transport of Reynolds Stress $u_iu_j$

- The Reynolds stresses appear as an unknown in the momentum equation.

- A transport equation for Reynolds stress $u_iu_j$ may be obtained as follows,

  Multiply the equation for the transport of turbulent velocity $u_i$ by $u_j$ and take time average.

  Add to the above equation that is identical except that indices $i$ and $j$ are interchanged, i.e.,

  \[
  \frac{\partial u_iu_j}{\partial t} = u_j \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_j}{\partial t}
  \]

- The stress transport equation may thus be written:

  \[
  \frac{\partial u_iu_j}{\partial t} + U_k \frac{\partial u_iu_j}{\partial x_k} = -\left(\frac{u_iu_k}{u_j} \frac{\partial U_j}{\partial x_k} + \frac{u_ju_k}{u_i} \frac{\partial U_i}{\partial x_k}\right) + \frac{p}{\rho} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)
  \]

  \[
  - \frac{\partial}{\partial x_k} \left[ \frac{u_iu_j u_k}{\rho} \delta_{ij} + \frac{u_j^2}{\rho} \delta_{ij} - \nu \left( \frac{\partial u_iu_j}{\partial x_k} + \frac{\partial u_ju_i}{\partial x_k} + \frac{\partial u_k}{\partial x_k} \right) \right] - 2\nu \left( \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} - \nu \left( \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_k} + \frac{\partial u_j}{\partial x_k} \frac{\partial u_k}{\partial x_k} \right) \right)
  \]

  \[
  \text{convection} - C_{ij} \quad \text{production} - P_{ij} \quad \text{redistribution} - \phi_{ij} \quad \text{diffusion} - d_{ij} \quad \text{dissipation} - c_{ij}
  \]
Pure Shear Flow

- Assuming that $U_1 = U_1(x_2), U_2 = U_3 = 0, U_j \partial/\partial x_j = 0, \partial/\partial x_1 = \partial/\partial x_3 = 0$.

- In this flow, the only nonzero component of $\partial U_i/\partial x_j$ is $\partial U_1/\partial x_2$, so that the only nonzero component of $S_{ij}$ are $S_{12}$ and $S_{21}$, both of which are equal to $(\partial U_1/\partial x_2)/2$.

- If the Reynolds number is large, the turbulent kinetic energy budget reads,

$$0 = -u_1 u_2 \frac{\partial U_1}{\partial x_1} - \frac{1}{\rho} \frac{\partial}{\partial x_2} \left( \frac{1}{2} u_2^2 + \frac{1}{2} u_1 u_2 \right) - \epsilon$$

All of these terms are of order $u^3/\epsilon$.

- It is worthwhile to compare the kinetic energy equation with the equations for the kinetic energy of the three velocity fluctuations individually.

- If the Reynolds number is large that dissipative structure can be assumed to be isotropic, the equations for $\overline{u_1^2}/2, \overline{u_2^2}/2, \overline{u_3^2}/2$ are,

$$0 = -\frac{u_1 u_2}{\rho} \frac{\partial U_1}{\partial x_2} + \frac{1}{\rho} \frac{\partial}{\partial x_1} \left( \frac{1}{2} \overline{u_1^2 u_2} \right) - \frac{1}{3} \epsilon,$$

$$0 = 0 + \frac{1}{\rho} \frac{\partial}{\partial x_2} \left( \frac{1}{2} \overline{u_2^2} u_2 \right) - \frac{1}{3} \epsilon,$$

$$0 = 0 + \frac{1}{\rho} \frac{\partial}{\partial x_3} \left( \frac{1}{2} \overline{u_3^2} u_3 \right) - \frac{1}{3} \epsilon.$$

The sum of these equations equals the kinetic energy equation, $\overline{u_i u_i}/2$. 
- Because of incompressibility,

\[
p\frac{\partial u_1}{\partial x_1} + p\frac{\partial u_2}{\partial x_2} + p\frac{\partial u_3}{\partial x_3} = p\frac{\partial u_i}{\partial x_i} = 0
\]

Because the sum of the pressure terms is equal to zero, the pressure terms exchange energy between components without changing the total amount of energy.

- If \(\overline{u_2^2}/2\) and \(\overline{u_3^2}/2\) which have no production terms, are to maintain themselves, notwithstanding dissipative losses, \(p\partial u_2/\partial x_2\) and \(p\partial u_3/\partial x_3\) must be positive, so that \(p\partial u_1/\partial x_1\) must be negative.

- In most flows \(\overline{u_1^2}/2\) is roughly twice as large as \(\overline{u_2^2}/2\) and \(\overline{u_3^2}/2\).

- In summary, the \(u_1\) component has more energy than the other components, because it receives all of the production of kinetic energy; the transfer of energy to the other components is performed by nonlinear pressure-velocity interactions.
Vorticity Dynamics

- All turbulent flows are characterized by high levels of fluctuating vorticity. This is the feature that distinguishes turbulence from other random fluid motions like ocean waves and atmospheric gravity waves.

- Reynolds stresses may be associated with eddies whose vorticity is roughly aligned with the mean strain rate.

- We then turn to a study of the vorticity equation. We shall find that vorticity can indeed be amplified by line stretching due to the strain rate.

- The equation for the mean vorticity in a turbulent shear flow also will be explored; the interactions between velocity and vorticity fluctuations again include both transport and stretching.
Because the scale of eddies that are stretched by a strain rate decreases, the energy transfer from large eddies to small eddies may be considered in terms of vortex stretching. We shall study the mean-square vorticity fluctuations $\langle \omega_i \omega_i \rangle$ in detail.

The ultimate energy transfer, the dissipation of kinetic energy into heat, will turn out to be approximately equal to $\nu \omega_i \omega_i$ if the Re is large.

In summary, this section attempts to explain what we mean when we say that turbulence is rotational and dissipative.
Vorticity Vector and Rotation Tensor

- The vorticity is the curl of the velocity vector:
  \[
  \tilde{\omega}_i = \varepsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial x_j}
  \]
e.g. \[
  \tilde{\omega}_3 = \frac{\partial \tilde{u}_2}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_2}, \quad \tilde{\omega}_2 = \frac{\partial \tilde{u}_1}{\partial x_3} - \frac{\partial \tilde{u}_3}{\partial x_1}
  \]
  where \( \varepsilon_{ijk} \) is an alternating tensor, it is +1 if i, j, k are in cyclic order, -1 if I, j, k are in anticyclic order, 0 if any two of i, j, k are equal.

- This relation shows that \( \tilde{\omega}_i \) is related to the deformation rate \( \frac{\partial \tilde{u}_i}{\partial x_j} \). The deformation rate can be split up into a symmetric and a skew-symmetric part;
  \[
  \frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{s}_{ij} + \tilde{r}_{ij}, \quad \tilde{s}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right), \quad \tilde{r}_{ij} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} \right)
  \]
  where the symmetric tensor \( \tilde{s}_{ij} \) is called the strain rate tensor, the skew–symmetric tensor \( \tilde{r}_{ij} \) is called the rotation tensor.
Sine the alternating tensor $\varepsilon_{ijk} = (-\varepsilon_{ikj})$ is a skew-symmetric tensor, the vorticity vector is related only to the skew-symmetric part of $\partial \tilde{u}_i / \partial x_j$

$$\tilde{\omega}_i = \varepsilon_{ijk} \tilde{r}_{kj}$$

since

$$\varepsilon_{ijk} \tilde{s}_{kj} = \frac{1}{2} \varepsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial x_j} + \frac{1}{2} \varepsilon_{ijk} \frac{\partial \tilde{u}_j}{\partial x_k} = \frac{1}{2} \varepsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial x_j} + \frac{1}{2} \varepsilon_{ikj} \frac{\partial \tilde{u}_k}{\partial x_j}$$

$$= \frac{1}{2} \varepsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial x_j} - \frac{1}{2} \varepsilon_{ijk} \frac{\partial \tilde{u}_k}{\partial x_j} = 0$$

Conversely, with some tensor algebra it is found that

$$\tilde{r}_{ij} = -\frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_k$$
Vortex Terms in the Equation of Motion

- The vorticity equation is obtained by taking the curl of the Navier-Stokes equations.

\[
\frac{\partial \tilde{u}_i}{\partial t} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial}{\partial x_j} (\tilde{u}_j \tilde{u}_i) + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}. \tag{3.3.6}
\]

The convective stress term may be decomposed as follows:

\[
\frac{\partial}{\partial x_j} (\tilde{u}_i \tilde{u}_j) = \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{u}_j \left( \frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} \right) + \tilde{u}_j \frac{\partial \tilde{u}_j}{\partial x_i}
\]

\[
= 2 \tilde{u}_j \tilde{r}_{ij} + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \tilde{u}_j \tilde{u}_j \right)
\]

\[
= -\epsilon_{ijk} \tilde{u}_j \tilde{\omega}_k + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \tilde{u}_j \tilde{u}_j \right). \tag{3.3.7}
\]
The viscous term may be expressed in terms of vorticity by putting

\[ \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} \right) + \nu \frac{\partial}{\partial x_i} \left( \frac{\partial \tilde{u}_i}{\partial x_j} \right) \]

\[ = 2 \nu \frac{\partial}{\partial x_j} \tilde{r}_{ij} + 0 \]

\[ = -\nu \epsilon_{ijk} \frac{\partial \tilde{\omega}_k}{\partial x_j}. \quad (3.3.8) \]

The continuity equation \( \partial u_i / \partial x_i = 0 \) was used to remove the second term.

If (3.3.7) and (3.3.8) are substituted into (3.3.6), there results

\[ \frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial}{\partial x_i} \left( \frac{\tilde{\rho}}{\rho} + \frac{1}{2} \tilde{u}_j \tilde{u}_j \right) + \epsilon_{ijk} \tilde{u}_j \tilde{\omega}_k - \nu \epsilon_{ijk} \frac{\partial \tilde{\omega}_k}{\partial x_j}. \quad (3.3.9) \]

In irrotational flow, \( \tilde{\omega}_k = 0 \) by definition, so that the viscous term and the vorticity part of the inertia term vanish. The inertia term then reduces to the gradient of the dynamic pressure \( \frac{1}{2} \rho \tilde{u}_j \tilde{u}_j \) and (3.3.9) reduces to the Bernoulli equation. In turbulent flow, of course, neither of these conditions is satisfied.
The cross-product term $\epsilon_{ijk} \tilde{u}_j \tilde{\omega}_k$ is crucial to turbulence theory. It is analogous to the Coriolis force $2 \epsilon_{ijk} \tilde{u}_j \tilde{\Omega}_k$ that would appear in the equation of motion if the coordinate system were rotating with an angular velocity $\tilde{\Omega}_k$ (the factor 2 is absent from the vorticity term because $\tilde{\omega}_k$ is twice the angular velocity of a small fluid element). The vortex term is also related to the lift force [Magnus effect] experienced by a vortex line exposed to a velocity $\tilde{u}_j$. A graphical interpretation of the "vortex force" may be helpful. In the equation for $\tilde{u}_1$, the term $\epsilon_{ijk} \tilde{u}_j \tilde{\omega}_k$ becomes $\tilde{u}_2 \tilde{\omega}_3 - \tilde{u}_3 \tilde{\omega}_2$. Figure 3.4 illustrates the geometry involved.
Reynolds Stress and Vorticity

- The instantaneous vorticity $\tilde{\omega}_i$ is decomposed into a mean vorticity $\Omega_i$ and vorticity fluctuation $\omega_i$:

$$\tilde{\omega}_i = \Omega_i + \omega_i, \quad \bar{\omega}_i = 0.$$ (3.3.10)

If we assume that the flow is steady in the mean, so that we can use time averages, the equation for the mean velocity $U_j$ may be written as

$$0 = -\frac{\partial}{\partial x_j} \left( \frac{\rho}{2} U_j U_j + \frac{1}{2} \overline{u_j u_j} \right) + \varepsilon_{ijk} (U_j \Omega_k + \overline{u_j \omega_k}) + \nu \frac{\partial^2 U_j}{\partial x_j \partial x_j}.$$ (3.3.11)

Clearly, Reynolds-stress gradients contain both a dynamic-pressure gradient and an interaction term between the vorticity fluctuations and the velocity fluctuations. In many turbulent flows the contribution of the turbulence to the dynamic pressure is insignificant because $\frac{1}{2} \overline{u_j u_j} \ll \frac{1}{2} U_j U_j$. The dynamic significance of the Reynolds stress is then associated mainly with the interaction between velocity and vorticity. For a closer look at this interaction, let us consider a two-dimensional mean flow in which $U_1 \gg U_2, U_3 = 0$, and in which downstream derivatives of mean quantities are small compared to cross-stream derivatives ($\partial/\partial x_1 \ll \partial/\partial x_2$). This corresponds to most boundary-layer and wake flows (see Chapters 4 and 5). Under these conditions, the only nonzero component of $\Omega_i$ is $\Omega_3 = \partial U_2 / \partial x_1 - \partial U_1 / \partial x_2$. Because $U_2 \ll U_1$ and $\partial/\partial x_1 \ll \partial/\partial x_2$, the vorticity component $\Omega_3$ is approximately equal to $-\partial U_1 / \partial x_2$. 
In the equation for $U_1$ the vorticity cross-product terms associated with the mean flow are $U_2 \Omega_3$ and $-U_3 \Omega_2$. The first of these is equal to $-U_2 \partial U_1 / \partial x_2 + U_2 \partial U_2 / \partial x_1$, the second is zero because $U_3 = 0$, $\Omega_2 = 0$. Also, $-\partial(\frac{1}{2} U_j U_j) / \partial x_1$ is equal to $-U_1 \partial U_1 / \partial x_1 - U_2 \partial U_2 / \partial x_1$ in this flow; the small term $U_2 \partial U_2 / \partial x_1$ cancels the same term generated by $U_2 \Omega_3$. If we neglect the viscous term and the contribution of the turbulence to the dynamic pressure, the equation for $U_1$ may be written as

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x_1} + u_2 \omega_3 - u_3 \omega_2. \quad (3.3.12)$$

Comparing (2.1.23) and (3.3.12) and observing that $\partial (\bar{u}_1 \bar{u}_2) / \partial x_2 \ll \partial (u_1 u_2) / \partial x_2$, we find that the vortex terms represent the cross-stream derivative of the Reynolds shear stress $-\bar{u}_1 \bar{u}_2$:

$$\frac{\partial}{\partial x_2} (-\bar{u}_1 \bar{u}_2) = u_2 \omega_3 - u_3 \omega_2. \quad (3.3.13)$$

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{1}{\rho} \frac{\partial P}{\partial x_1} + \frac{\partial}{\partial x_1} (-\bar{u}_1 \bar{u}_1) + \frac{\partial}{\partial x_2} (-\bar{u}_1 \bar{u}_2) \quad (2.1.23)$$
Some understanding of the turbulent vorticity terms in (3.3.13) may be obtained by employing the estimate

$$-u_1 u_2 \sim u \ell \partial U_1 / \partial x_2.$$  \hspace{1cm} (3.3.14)

If $u$ is approximately independent of $x_2$ (this is true for many turbulent shear flows), the Reynolds-stress gradient becomes

$$\frac{\partial}{\partial x_2} \left( -u_1 u_2 \right) \sim u \ell \frac{\partial^2 U_1}{\partial x_2^2} + u \frac{\partial \ell}{\partial x_2} \frac{\partial U_1}{\partial x_2}.$$  \hspace{1cm} (3.3.15)

Of course, (3.3.15) needs to be viewed with considerable reservation because (3.3.14) is a scaling law, not an equation. Because $\partial U_1 / \partial x_2 = -\Omega_3$ approximately, (3.3.15) may be written as

$$\frac{\partial}{\partial x_2} \left( -u_1 u_2 \right) \sim -u \ell \frac{\partial \Omega_3}{\partial x_2} - u \Omega_3 \frac{\partial \ell}{\partial x_2}.$$  \hspace{1cm} (3.3.16)
Let us now consider $u_2 \omega_3$ and $u_3 \omega_2$. In the flow treated here, the only nonzero component of $\Omega_i$ is $\Omega_3$. If vorticity can be transported in the $x_2$ direction by $u_2$ in the same way as momentum is transported, we should be able to write

$$u_2 \omega_3 \sim -u \partial \Omega_3 / \partial x_2.$$  (3.3.17)

The adoption of this expression constitutes a mixing-length theory of vorticity transfer (Taylor, 1932).

From a comparison of (3.3.13) and (3.3.16) we conclude that the nature of $u_3 \omega_2$ is associated with a change-of-scale effect:

$$u_3 \omega_2 \sim u \Omega_3 \frac{\partial \ell}{\partial x_2}.$$  (3.3.18)

The term $u_3 \omega_2$ may be called a vortex-stretching force, since it is associated with the change of size of eddies with vorticity of order $\Omega_3$ (see also the discussion following (3.3.35)).
The relative contributions of $u_2 \omega_3$ and $u_3 \omega_2$ to $\delta(-u_1 u_2)/\delta x_2$ apparently depend on the kind of flow considered. If the length scale $\ell$ is approximately constant across the flow, the vortex-stretching force (3.3.18) should be negligible; the Reynolds-stress gradient may then be interpreted as vorticity transport, which should scale according to (3.3.17). This may explain why vorticity transport theory has had some success in the description of turbulent wakes and jets: in those flows, the length scale is roughly constant in the cross-stream direction.
If the length scale $\ell$ changes in the $x_2$ direction, vorticity transport theory is inadequate. A case in point is the surface layer with constant stress ($-u_1u_2 = u_*^2$). In this flow,

\[ -u_1u_2 = u_*^2 = \kappa x_2 u_* \frac{\partial U_1}{\partial x_2}, \]  

so that

\[ \frac{\partial}{\partial x_2} (-u_1u_2) = 0 = \kappa x_2 u_* \frac{\partial^2 U_1}{\partial x_2^2} + \kappa u_* \frac{\partial U_1}{\partial x_2}. \]  

(3.3.19)

According to (3.3.19), $\partial U_1/\partial x_2 = u_*/\kappa x_2$, so that $\partial^2 U_1/\partial x_2^2 < 0$. The vorticity-transport term $\kappa x_2 u_* \frac{\partial^2 U_1}{\partial x_2^2} = -\kappa x_2 u_* \frac{\partial \Omega_3}{\partial x_2}$ thus is a deceleration. The deceleration of this flow is avoided because the vortex-stretching force $\kappa u_* \frac{\partial U_1}{\partial x_2} = -\kappa u_* \Omega_3$ balances the vorticity-transport force.
One final observation needs to be made. If the local length scale of the mean-flow field is comparable to the eddy size $\ell$, the order of magnitude of $\overline{u_2\omega_3}$ and $\overline{u_3\omega_2}$ is $u^2/\ell$. Now, as we see later in this section, $\omega_j$ is of order $u/\lambda$, so that the correlation coefficient between $\omega_j$ and $u_j$ is of order $\lambda/\ell$. This is similar to the correlation between $u_i$ and $s_{ij}$ which was discussed earlier; the correlation is poor because most contributions to $\omega_j$ are made at high frequencies while most of $u_j$ is associated with low frequencies.

\[
\frac{\lambda}{\ell} = \left( \frac{15}{A} \right)^{1/2} \left( \frac{u\ell}{\nu} \right)^{-1/2} = \left( \frac{15}{A} \right)^{1/2} R_{\ell}^{-1/2} = \frac{15}{A} \left( \frac{u\lambda}{\nu} \right)^{-1} = \frac{15}{A} R_{\lambda}^{-1}
\]
The Vorticity Equation

- The vorticity equation is obtained by applying the operator “curl” ($\epsilon_{pq i} \partial / \partial x_q$) to the Navier-Stokes equation:

$$\frac{\partial \tilde{\omega}_p}{\partial t} = -\epsilon_{pq i} \frac{\partial^2}{\partial x_i \partial x_q} \left( \frac{\tilde{p}}{\rho} + \frac{1}{2} \tilde{u}_j \tilde{u}_j \right)$$

$$+ (\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) \left( \frac{\partial}{\partial x_q} \tilde{u}_j \tilde{\omega}_k - \nu \frac{\partial^2 \tilde{\omega}_k}{\partial x_q \partial x_j} \right).$$

(3.3.21)

Here, the tensor identity $\epsilon_{pq i} \epsilon_{ij k} = \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}$ has been used. The pressure term in (3.3.21) is zero because it involves the product of the skew-symmetric tensor $\epsilon_{pq i}$ and the symmetric tensor operator $\partial^2 / \partial x_i \partial x_q$. 
Accounting for all of the Kronecker deltas in (3.3.21), we obtain

\[
\frac{\partial \tilde{\omega}_p}{\partial t} = \tilde{\omega}_k \frac{\partial \tilde{u}_p}{\partial x_k} - \tilde{u}_k \frac{\partial \tilde{\omega}_p}{\partial x_k} - \nu \frac{\partial}{\partial x_p} \left( \frac{\partial \tilde{\omega}_k}{\partial x_k} \right) + \nu \frac{\partial^2 \tilde{\omega}_p}{\partial x_k \partial x_k}.
\]  

(3.3.22)

The first of the viscous terms in (3.3.22) is zero because vorticity has zero divergence (the divergence of the curl of a vector is zero):

\[
\frac{\partial \tilde{\omega}_k}{\partial x_k} = \varepsilon_{ijk} \frac{\partial^2 \tilde{u}_j}{\partial x_i \partial x_k} = 0.
\]  

(3.3.23)

The final form of the vorticity equation is (changing \( p \) to \( i \) and the dummy index \( k \) to \( j \) for convenience)

\[
\frac{\partial \tilde{\omega}_j}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_j}{\partial x_j} = \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j \partial x_j}.
\]  

(3.3.24)
Before we interpret the first term on the right-hand side of (3.3.24), we want
to show that the skew-symmetric part \( \tilde{r}_{ij} \) of \( \partial \tilde{u}_i / \partial x_j \) does not contribute to it.
For this purpose, \( \partial \tilde{u}_i / \partial x_j \) is split up into \( \tilde{r}_{ij} \) and \( \tilde{s}_{ij} \), such that

\[
\tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{\omega}_j \tilde{s}_{ij} + \tilde{\omega}_j \tilde{r}_{ij}. 
\]  
(3.3.25)

Because of the definition of \( \tilde{r}_{ij} \), the second term in (3.3.25) becomes

\[
\tilde{\omega}_j \tilde{r}_{ij} = -\frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k. 
\]  
(3.3.26)

Since \( j \) and \( k \) are dummy indices they may be interchanged to yield

\[
-\frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k = -\frac{1}{2} \varepsilon_{ikj} \tilde{\omega}_j \tilde{\omega}_k. 
\]  
(3.3.27)

Again interchanging the indices \( j \) and \( k \) in \( \varepsilon_{ikj} \), we obtain a change in sign
because \( \varepsilon_{ijk} \) is skew-symmetric. Hence, we find

\[
-\frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k = \frac{1}{2} \varepsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k. 
\]  
(3.3.28)

This can be true only if this term is zero.
The vorticity equation then may be written as
\[
\frac{\partial \tilde{\omega}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} = \tilde{\omega}_j \tilde{s}_{ij} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j \partial x_j}.
\] (3.3.29)

The term \(\tilde{\omega}_j \tilde{s}_{ij}\) represents amplification and rotation of the vorticity vector by the strain rate. In the context of this section, the turning of vortex axes by the strain rate is of minor importance; we shall concentrate on the components of \(\tilde{\omega}_j \tilde{s}_{ij}\) that represent vortex stretching.

Vorticity apparently can be amplified by stretching of present vorticity by the strain rate \(\tilde{s}_{ij}\). On the other hand, vorticity is decreased in an environment where squeezing \((\tilde{s}_{ij} < 0)\) occurs.

This "source" or "sink" for vorticity is the most interesting term of the vorticity equation. It is essential to recognize that the term does not occur in two-dimensional flow. Suppose a flow is entirely in the \(x_1, x_2\) plane. Then \(\tilde{\omega}_1\) and \(\tilde{\omega}_2\) are zero, so that the only nonzero vorticity component is \(\tilde{\omega}_3\). The vortex-stretching term then becomes \(\tilde{\omega}_3 \tilde{s}_{i3}\). However, in a two-dimensional flow only \(\tilde{s}_{12} (=\tilde{s}_{21})\), \(\tilde{s}_{11}\), and \(\tilde{s}_{22}\) can be different from zero. A two-dimensional flow cannot turn or stretch the vorticity vector.
A simple illustration of vortex stretching is the accelerated flow in a wind-tunnel contraction. Here (Figure 3.5) \( \tilde{s}_{11} \) is positive, so that \( \tilde{s}_{22} \) and \( \tilde{s}_{33} \) must be negative to satisfy the continuity equation \( (\tilde{s}_{ii} = 0) \). In this kind of flow, \( \tilde{\omega}_1 \) is increased by vortex stretching, while \( \tilde{\omega}_2 \) and \( \tilde{\omega}_3 \) are attenuated.

Figure 3.5. Vortex stretching in a wind-tunnel contraction. As the flow speeds up from left to right, the vorticity component \( \omega_1 \) is amplified because angular momentum has to be conserved.
The change of vorticity by vortex stretching is a consequence of the conservation of angular momentum. The angular momentum of a material volume element would remain constant if viscous effects were absent; if the fluid element is stretched so that its cross-sectional area and moment of inertia become smaller, the component of the angular velocity in the direction of the stretching must increase in order to conserve angular momentum. Vortex stretching always involves a change of length scale, as Figure 3.5 illustrates.

\[
\frac{\partial \tilde{u}_1}{\partial x_1} + \frac{\partial \tilde{u}_2}{\partial x_2} + \frac{\partial \tilde{u}_3}{\partial x_3} = 0
\]

\[
\tilde{\omega}_1 : \tilde{\omega}_1 \frac{\partial \tilde{u}_1}{\partial x_1}
\]

\[
\tilde{\omega}_2 : \tilde{\omega}_2 \frac{\partial \tilde{u}_2}{\partial x_2}
\]

\[
\tilde{\omega}_3 : \tilde{\omega}_3 \frac{\partial \tilde{u}_3}{\partial x_3}
\]
Vorticity in Turbulent Flows

- In turbulent flow, the vorticity is decomposed into a mean vorticity $\Omega_i$ and vorticity fluctuation $\omega_i$ according to (3.3.10). After substituting (3.3.10) and the corresponding Reynolds decompositions for $\tilde{u}_i$ and $\tilde{s}_{ij}$ into (3.3.29) and taking the average of all terms in the equation, we obtain

$$U_j \frac{\partial \Omega_i}{\partial x_j} = -u_j \frac{\partial \omega_i}{\partial x_j} + \omega_j \tilde{s}_{ij} + \Omega_j \tilde{s}_{ij} + \nu \frac{\partial^2 \Omega_i}{\partial x_j \partial x_j}.$$  

(3.3.30)

The mean flow has been assumed to be steady.
From (3.3.10) and (3.3.23) we conclude that both the mean vorticity and the fluctuating vorticity are solenoidal (that is, divergenceless):

$$\frac{\partial \Omega_j}{\partial x_i} = 0, \quad \frac{\partial \omega_i}{\partial x_i} = 0.$$  \hspace{1cm} (3.3.31)

With the second equation in (3.3.31) and the continuity equation $\partial u_j/\partial x_j = 0$, the turbulence terms in (3.3.30) can be rearranged as follows:

$$u_j \frac{\partial \omega_i}{\partial x_j} = \frac{\partial}{\partial x_j} (u_j \omega_i),$$  \hspace{1cm} (3.3.32)

$$\omega_j s_{ij} = \omega_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} (\omega_j u_i).$$  \hspace{1cm} (3.3.33)

The term given in (3.3.32) is clearly analogous to the Reynolds-stress term in the equation for $U_j$; it is due to mean transport of $\omega_i$ through its interaction with fluctuating velocities $u_j$ in the direction of the gradients $\partial/\partial x_j$. This term, of course, changes the mean vorticity only if $\overline{u_j \omega_i}$ changes in the $x_j$ direction. Properly speaking, (3.3.32) is a transport "divergence."

The term given in (3.3.33) is the gain (or loss) of mean vorticity caused by the stretching and rotation of fluctuating vorticity components by fluctuating strain rates.
Two-Dimensional Mean Flow

In a flow with \( U_3 = 0, \Omega_1 = \Omega_2 = 0, \partial / \partial x_3 = 0, \)
and \( \partial / \partial x_1 \ll \partial / \partial x_2 \) (whose equation of motion was discussed earlier), the
major turbulence terms in the equation for \( \Omega_3 \) are

\[
\bar{u}_j \frac{\partial \bar{\omega}_3}{\partial x_j} \approx \frac{\partial}{\partial x_2} (\bar{u}_2 \bar{\omega}_3),
\]

(3.3.34)

\[
\bar{\omega}_j s_{3j} \approx \frac{\partial}{\partial x_2} (\bar{u}_3 \bar{\omega}_2).
\]

(3.3.35)

The products \( \bar{u}_2 \bar{\omega}_3 \) and \( \bar{u}_3 \bar{\omega}_2 \) are related to the Reynolds-stress gradient by
(3.3.13); \( \bar{u}_2 \bar{\omega}_3 \) was interpreted as a body force arising from transport of \( \bar{\omega}_3 \)
by \( \bar{u}_2 \) in a field with a mean gradient \( \partial \Omega_3 / \partial x_2 \), whereas \( \bar{u}_3 \bar{\omega}_2 \) was interpreted
as a body force associated with the change of size of eddies in a flow field
with a varying length scale. The vortex-stretching nature of \( \bar{u}_3 \bar{\omega}_2 \) is con-
irmed by (3.3.35). The cross-stream gradients of these body forces are
sources or sinks for mean vorticity. In a surface layer with constant stress, the
mean vorticity \( \Omega_3 \) is constant along streamlines; from (3.3.17, 3.3.34) and
(3.3.18, 3.3.35) we may conclude that \( \Omega_3 \) is maintained because the gain of
mean vorticity due to a net transport surplus is balanced by the loss of mean
vorticity due to the transfer of vorticity to the turbulence by vortex stretch-
ing.
A more comprehensive interpretation of (3.3.34) and (3.3.35) becomes extremely involved. Even if (3.3.17) and (3.3.18) are adopted as crude models of $u_2 \omega_3$ and $u_3 \omega_2$, respectively, it would be presumptuous to differentiate these equations in order to obtain models for (3.3.34, 3.3.35), because that would amount to differentiating the Reynolds-stress scaling law (3.3.14) twice. In vorticity-transfer theory, of course, the term $\omega_j s_{ij}$ is ignored and the transport term (3.3.34) is scaled on basis of (3.3.17).

In the discussion following (3.3.20) we found that $u_2 \omega_3$ and $u_3 \omega_2$ both are of order $u^2/\ell$. The cross-stream gradients ($\partial/\partial x_2$) should scale with the local length scale of the mean flow, which is comparable to $\ell_i$ in flows without multiple scales. Therefore, (3.3.34) and (3.3.35) are of order $u^2/\ell^2$. 
The Dynamics of $\Omega_i\Omega_i$

An equation for the square of the mean vorticity is needed because the interaction between mean and fluctuating vorticities can be studied only in terms of $\Omega_i\Omega_i$ and $\overline{\omega_j\omega_j}$. Multiplying (3.3.30) by $\Omega_i$ and rearranging terms, we find

$$U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \Omega_i\Omega_i \right) = -\frac{\partial}{\partial x_j} (\Omega_i \omega_j u_j) + u_j \omega_i \frac{\partial \Omega_i}{\partial x_j} + \Omega_i \Omega_j s_{ij}$$

$$+ \Omega_i \omega_j s_{ij} + \nu \frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{1}{2} \Omega_i\Omega_i \right) - \nu \frac{\partial \Omega_i}{\partial x_j} \frac{\partial \Omega_i}{\partial x_j}.$$  

(3.3.36)

The first term on the right-hand side of (3.3.36) is the transport of $\Omega_i\Omega_i$ by turbulent vorticity-velocity interactions. This term is equivalent to the turbulent transport term of $U_iU_i$. The second term on the right-hand side of (3.3.36) is like the turbulence-production term in the energy equation. We may call it gradient production of $\overline{\omega_j\omega_j}$ in anticipation of the occurrence of the same term (with opposite sign) in the equation for $\overline{\omega_j\omega_j}$. The third term is stretching or shrinking of mean vorticity by the mean strain rate. The fourth term is amplification or attenuation of $\Omega_i\Omega_i$ caused by the stretching of fluctuating vorticity components by fluctuating strain rates. The fifth term is viscous transport of $\Omega_i\Omega_i$, and the sixth is viscous dissipation of $\Omega_i\Omega_i$. 
The mean vorticity $\Omega_j$ is of order $u/l$. Because $\omega_j u_j \sim u^2/l$ and $\omega_j s_{ij} \sim u^2/l^2$, the viscous terms in (3.3.36) are of order $(u^3/l^3) (v/u)$, and all the other terms are of order $u^3/l^3$. Generally speaking, therefore, only the viscous terms can be neglected. In a two-dimensional flow in the $x_1, x_2$ plane the only nonzero component of $\Omega_j$ is $\Omega_3$. At large Reynolds numbers, (3.3.36) may then be approximated by

$$U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \Omega_3 \Omega_3 \right) = - \frac{\partial}{\partial x_j} \left( \Omega_3 \omega_3 u_j \right) + u_j \omega_3 \frac{\partial \Omega_3}{\partial x_j} + \Omega_3 \omega_j s_{3j}. \quad (3.3.37)$$

The stretching term $\Omega_i \Omega_j s_{ij}$ is zero in two-dimensional flow. If the flow involves no change of length scale, the last term of (3.3.37) may be neglected (see the discussion following (3.3.35)).
The Equation for $\omega_i \omega_i$

The equation of the mean-square vorticity fluctuations is obtained by a procedure exactly similar to the one followed for the equation of the turbulent kinetic energy. We leave the algebra as an exercise for the reader; the final result is, if the flow is steady in the mean,

$$U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \omega_i \omega_i \right) = -u_j \omega_i \frac{\partial \Omega_i}{\partial x_j} - \frac{1}{2} \frac{\partial}{\partial x_j} (u_j \omega_i \omega_i) + \omega_i \omega_j s_{ij} + \omega_i \omega_j S_{ij}$$

$$+ \Omega_j \omega_i s_{ij} + \nu \frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{1}{2} \omega_i \omega_i \right) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}. \quad (3.3.38)$$

The first term on the right-hand side of (3.3.38) is the gradient production of $\omega_i \omega_i$. This term exchanges vorticity between $\omega_i \omega_i$ and $\Omega_i \Omega_i$ in the same way as turbulent energy production $(-\overline{u_i u_j} S_{ij})$ exchanges energy between $U_i U_i$ and $u_i u_i$.

The second term is the transport of mean-square turbulent vorticity by turbulent velocity fluctuations. This term is analogous to the transport term $\partial (u_i u_i u_j) / \partial x_j$ in the equation for $u_i u_i$. 
\[ U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \Omega_i \Omega_i \right) = -u_j \omega_i \frac{\partial \Omega_j}{\partial x_j} - \frac{1}{2} \frac{\partial}{\partial x_j} \left( u_j \omega_i \omega_i \right) + \omega_i \omega_j s_{ij} + \omega_i \omega_j S_{ij} \]

\[ + \Omega_j \omega_i s_{ij} + \nu \frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{1}{2} \omega_i \omega_i \right) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} \]  \quad (3.3.38)

The third term is the production of mean-square turbulent vorticity by turbulent stretching of turbulent vorticity. We shall soon see that this is one of the dominant terms in the equation for \( \Omega_i \Omega_i \).

The fourth term is the production (or removal, as the case may be) of turbulent vorticity caused by the stretching (or squeezing) of vorticity fluctuations by the mean rate of strain \( S_{ij} \).

The fifth term is a mixed production term. It occurs in the equation for \( \Omega_i \Omega_i \) with the same sign. Evidently, the stretching of fluctuating vorticity by strain-rate fluctuations produces \( \Omega_i \Omega_i \) and \( \omega_i \omega_i \) at the same rate.

The sixth and seventh terms on the right-hand side of (3.3.38) are viscous transport and dissipation of \( \omega_i \omega_i \), respectively.
Turbulence is Rotational

The equation for $\bar{\omega}_i \bar{\omega}_j$ looks nearly intractable. However, if the Reynolds number is large, a very simple approximate form of (3.3.38) can be obtained, because strain-rate fluctuations are much larger than the mean strain rate and vorticity fluctuations are much larger than the mean vorticity:

\[
S_{ij}S_{ij} = \mathcal{O} \left( \frac{\nu}{\lambda} \right)^2, \quad S_{ij}S_{ij} = \mathcal{O} \left( \frac{\nu}{\ell} \right)^2,  
\]

\[
\bar{\omega}_i \bar{\omega}_j = \mathcal{O} \left( \frac{\nu}{\lambda} \right)^2, \quad \Omega_i \Omega_j = \mathcal{O} \left( \frac{\nu}{\ell} \right)^2.  
\]

As before, $\mathcal{O}$ stands for "order of magnitude." The estimates for $s_{ij}$, $S_{ij}$, and $\Omega_i \Omega_j$ were obtained earlier; we have to prove that the first of (3.3.40) is a valid statement before we can proceed.
Some tensor algebra applied to the definitions of $s_{ij}$, $r_{ij}$, and $\omega_i$ yields

\[
\omega_i \omega_i = 2 r_{ij} r_{ij},
\]

(3.3.41)

\[
s_{ij} s_{ij} - r_{ij} r_{ij} = \partial^2 (u_i u_j) / \partial x_i \partial x_j.
\]

(3.3.42)

Now, $s_{ij} s_{ij}$ is of order $u^2 / \lambda^2$, but the right-hand side of (3.3.42) is of order $u^2 / \ell^2$. Consequently, at large Reynolds numbers (3.3.42) is approximated by

\[
s_{ij} s_{ij} \approx r_{ij} r_{ij}.
\]

(3.3.43)

Substituting this into (3.3.41), we find

\[
\omega_i \omega_i \approx 2 s_{ij} s_{ij}.
\]

(3.3.44)

From this we conclude that $\omega_i$ is of order $u / \lambda$, just like $s_{ij}$. This proves that the first of (3.3.40) is a valid statement if the Reynolds number is large enough. Turbulence indeed is rotational, with large vorticity fluctuations.
The strain-rate fluctuations are associated with viscous dissipation of turbulent energy. We recall that the dissipation rate $\epsilon$ is defined by

$$\epsilon = 2\nu \sum s_{ij} s_{ij}. \quad (3.3.45)$$

Because of (3.3.44), this may be rewritten as

$$\epsilon \approx \nu \sum \omega_i \omega_i. \quad (3.3.46)$$

This relation shows that dissipation of energy is also associated with vorticity fluctuations. This is a useful result, but it should be kept in mind that a causal relation exists only between the strain-rate fluctuations and the dissipation rate. Indeed, (3.3.44) states merely that in flows with high Reynolds numbers the symmetric and skew-symmetric parts of the deformation-rate tensor have about the same mean-square value.
An Approximate Vorticity Budget

The estimates (3.3.39) and (3.3.40) should enable us to simplify the vorticity budget (3.3.38) appreciably. However, many of the terms in (3.3.38) contain mixed products like $\omega_j u_j$ and $\omega_j s_{ij}$, which have to be estimated with care because they are nonzero due to the distorting effect of the mean strain rate $S_{ij}$. From (3.3.13) we concluded before that

$$u_j \omega_j = O(u^2/l);$$  \hspace{1cm} (3.3.47)

from (3.3.13) and (3.3.33) we concluded that

$$\omega_j s_{ij} = O(u^2/l^2).$$  \hspace{1cm} (3.3.48)
We also need the orders of magnitude of $\omega_i \omega_j$ and of $u_i \omega_i \omega_j$. The diagonal components of $\omega_i \omega_j$ are of order $\alpha^2/\lambda^2$, but the off-diagonal components are different from zero only in response to a mean strain rate. The mean strain rate $S_{ij}$ is of order $\alpha/\ell$ so that it can only weakly affect the vorticity structure whose characteristic frequency is $\alpha/\lambda$. Therefore, we expect that the effect of $S_{ij}$ should be proportional to the time-scale ratio $(\lambda/\alpha)/(\ell/\alpha) = \lambda/\ell$:

$$\omega_i \omega_j = \frac{\alpha^2}{\lambda^2} \left( a \delta_{ij} + b_{ij} \frac{\lambda}{\ell} + \ldots \right).$$  \hspace{1cm} (3.3.49)

The coefficients $a$ and $b_{ij}$ should be of order one. The discount for the time-scale ratio $\lambda/\ell$ applied here is analogous to the discount needed in $u_i \omega_j$.

The term $\omega_i \omega_j S_{ij}$ in (3.3.38) becomes

$$\omega_i \omega_j S_{ij} = \frac{\alpha^2}{\lambda^2} \left( a S_{ii} + b_{ij} \frac{\lambda}{\ell} S_{ij} + \ldots \right).$$  \hspace{1cm} (3.3.50)

Because $S_{ii} = 0$ as a result of incompressibility, and $b_{ij} S_{ij} \sim \alpha/\ell$, we find that

$$\omega_i \omega_j S_{ij} = O\left( \frac{\alpha^3}{\lambda \ell^2} \right).$$  \hspace{1cm} (3.3.51)
The transport term $\frac{\partial (u_j \omega; \omega)}{\partial x_j}$ may be written as

$$
\frac{\partial}{\partial x_j} \left( u_j \omega; \omega \right) = u_j \frac{\partial}{\partial x_j} (\omega; \omega).
$$

(3.3.52)

This term does not depend on the mean strain rate but on inhomogeneity in the distribution of mean square vorticity. If we assume that turbulent motion is an effective "mixer" of vorticity, $u_j$ should be well correlated with the gradients of $\omega; \omega$, so that

$$
\frac{\partial}{\partial x_j} (u_j \omega; \omega) = O \left( \frac{u}{\ell} \cdot \frac{u^2}{\lambda^2} \right) = O \left( \frac{u^3}{\lambda^2 \ell} \right).
$$

(3.3.53)

With the results obtained above, most of the terms of (3.3.38) can be estimated. We obtain

$$
-u_j \omega; \frac{\partial \Omega_i}{\partial x_j} = O \left( \frac{u^2}{\ell} \cdot \frac{u}{\ell^2} \right) = O \left( \frac{u^3}{\lambda^3 \cdot \ell^3} \right),
$$

(3.3.54)

$$
\Omega_j \omega; s_{ij} = O \left( \frac{u}{\ell} \cdot \frac{u^2}{\ell^2} \right) = O \left( \frac{u^3}{\lambda^3 \cdot \ell^3} \right).
$$

(3.3.55)
\[ \nu \frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{1}{2} \frac{\omega_i}{\omega_i} \right) = \mathcal{O} \left( \frac{\nu}{\ell^2} \cdot \frac{u^2}{\lambda^2} \right) = \mathcal{O} \left( \frac{u^3}{\lambda^3} \cdot \frac{\lambda^3}{\ell^3} \right), \]  
(3.3.56)

\[ \frac{\omega_i}{\omega_j} S_{ij} = \mathcal{O} \left( \frac{u^2}{\lambda \ell} \cdot \frac{u^2}{\ell} \right) = \mathcal{O} \left( \frac{u^3}{\lambda^3} \cdot \frac{\lambda^2}{\ell^2} \right), \]  
(3.3.57)

\[ \frac{1}{2} \frac{\partial}{\partial x_j} \left( u_j \omega_i \omega_i \right) = \mathcal{O} \left( \frac{u^2}{\ell} \cdot \frac{u^2}{\lambda^2} \right) = \mathcal{O} \left( \frac{u^3}{\lambda^3} \cdot \frac{\lambda}{\ell} \right), \]  
(3.3.58)

\[ U_j \frac{\partial}{\partial x_j} \left( \frac{1}{2} \frac{\omega_i}{\omega_i} \right) = \mathcal{O} \left( \frac{u^2}{\ell} \cdot \frac{u^2}{\lambda^2} \right) = \mathcal{O} \left( \frac{u^3}{\lambda^3} \cdot \frac{\lambda}{\ell} \right), \]  
(3.3.59)

\[ \frac{\omega_i}{\omega_j} s_{ij} = \mathcal{O} \left( \frac{u^2}{\lambda^2} \cdot \frac{u}{\lambda} \right) = \mathcal{O} \left( \frac{u^3}{\lambda^3} \cdot 1 \right), \]  
(3.3.60)

\[ \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} = ?, \]  
(3.3.61)
In the stretching term (3.3.60), no prorating with $\lambda/\ell$ is necessary, because $\omega_i$ operates on the same time scale as $s_{ij}$. The viscous dissipation term (3.3.61) has been left undecided, since we expect dissipation of vorticity to occur mainly at length scales smaller than $\lambda$. In the viscous diffusion term (3.3.56), the relation $\ell^2/\lambda^2 \sim u\ell/v$ has been used. In the transport term (3.3.59), the operator $U_j \partial/\partial x_j$ has been estimated as $u/\ell$; that choice is consistent with the estimates used in the equations for the mean flow and the turbulent kinetic energy (see 3.2.28, 3.2.31, 3.2.32).

The expressions (3.3.54) through (3.3.60) have been arranged in increasing order of magnitude. If the Reynolds number is large, all of the terms (3.3.54) through (3.3.59) are smaller than the turbulent stretching term (3.3.60) by at least a factor of $\lambda/\ell$, which is of order $R_{\lambda}^{-1/2}$. Therefore, at sufficiently high Reynolds numbers the turbulent vorticity budget (3.3.38) may be approximated as (Taylor, 1938)

$$\omega_i \omega_j s_{ij} = \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_j}{\partial x_j}.$$  (3.3.62)
The budget of mean-square vorticity fluctuations is thus approximately independent of the structure of the mean flow. Turbulent vorticity fluctuations, unlike turbulent velocity fluctuations, do not need the continued presence of a source term associated with the mean flow field. Of course, in the absence of a source of energy, turbulent vorticity fluctuations will decay, too. Also, the rate of change of \( \bar{\omega} \), as represented by (3.3.59), is small compared to the rate at which turbulent vortex stretching occurs. In Chapter 8 it will be shown that these conclusions lead to the concept of an equilibrium spectrum of turbulence at small scales.
The right-hand side of (3.3.62) is quadratic in \( \partial \omega_i / \partial x_j \), so that it is always positive. Hence, the left-hand side is positive, too. This implies that, on the average, there is more turbulent vortex stretching than vortex squeezing: vortex stretching transfers turbulent vorticity and the energy associated with it) from large-scale fluctuations to small-scale fluctuations. In this way turbulence obtains the broad energy spectrum that is observed experimentally, and in this way the very smallest eddies (which suffer rapid viscous decay) are continually being supplied with new energy. The approximate vorticity budget (3.3.62) is just as essential to understanding turbulence dynamics as the approximate energy budget (3.2.6). The relationship between these two budgets, incidentally, is a close one: viscous dissipation of vorticity prevents vorticity production \( (\omega_i \omega_j S_{ij}) \) from increasing \( \omega_i \omega_i \) without limit; while viscous dissipation of energy (which is proportional to \( \omega_i \omega_i \)) prevents the energy production \( (-u_i u_j S_{ij}) \) from increasing \( u_i u_i \) without limit. Vortex stretching makes \( \omega_i \omega_i \) as large as viscosity will permit; at large Reynolds numbers the mean-square strain-rate fluctuations keep pace, so that the turbulent energy is subject to rapid dissipation.
Two points need to be emphasized. **First**, in two-dimensional "turbulence" there is no vortex stretching, so that the vorticity budget (3.3.62) is irrelevant in that case. This implies that the spectral energy-transfer concepts developed here do not apply to two-dimensional stochastic flow fields.

**Second**, vorticity amplification is a result of the kinematics of turbulence. As an example, take a situation in which the principal axes of the instantaneous strain rate are aligned with the coordinate system, so that $s_{ij}$ has only diagonal components ($s_{11}$, $s_{22}$, and $s_{33}$). Let us assume for simplicity that $s_{22} = s_{33}$, so that, by virtue of continuity, $s_{11} = -2s_{22}$. The term $\omega_i \omega_j s_{ij}$ becomes, if we also assume that $\omega_2^2 = \omega_3^2$,

$$\omega_1^2 s_{11} + \omega_2^2 s_{22} + \omega_3^2 s_{33} = s_{11}(\omega_1^2 - \omega_2^2).$$

(3.3.63)

If $s_{11} > 0$, $\omega_1^2$ is amplified (see Figure 3.4), but $\omega_2^2$ and $\omega_3^2$ are attenuated because $s_{22}$ and $s_{33}$ are negative. Thus, $\omega_1^2 - \omega_2^2$ tends to become positive if $s_{11}$ is positive. Again, if $s_{11} < 0$, $\omega_1^2$ decreases, but $\omega_2^2$ and $\omega_3^2$ increase, so that $\omega_1^2 - \omega_2^2 < 0$, making the stretching term positive again.
Multiple Length Scales

If the vorticity gradients $\partial \omega / \partial x_j$ in (3.3.62) were estimated as $u/\lambda^2$, the dissipation term would be smaller than the stretching term. However, $\lambda$ is not the proper length scale for estimates of $\omega_i$ and $u$ is not the proper velocity scale; all we know is that the ratio $u/\lambda$ is the order of magnitude of $\omega_i$. Clearly, we need a new length scale. Calling it $\delta$, using (3.3.60), and requiring that the two sides of (3.3.62) have the same order of magnitude, we obtain

$$\nu \frac{u^2}{\lambda^2 \delta^2} = \mathcal{O} \left( \frac{u^3}{\lambda^3} \right).$$

(3.3.64)

The ratio $\delta/\lambda$ becomes

$$\delta/\lambda = \mathcal{O} (\nu u \lambda)^{1/2} = \mathcal{O} (R_{\lambda}^{-1/2}).$$

(3.3.65)
Comparing this with (3.2.18), we see that $\delta$ is proportional to the Kolmogorov microscale $\eta$. The Kolmogorov microscale thus has a role in the turbulent vorticity budget which is comparable to the role of the Taylor microscale in the turbulent energy budget. Since vortex stretching is the only known spectral energy-transfer mechanism, $\eta$ is the smallest length scale possible: the dynamics of $\left(\frac{\partial \omega_i}{\partial x_j}\right)^2$ would not lead to a length scale smaller than $\eta$.

Since the vorticity budget is approximately independent of the structure of the mean flow, vorticity dynamics can be studied more easily in the wave-number (spectral) domain than in the spatial domain.

$$\frac{\lambda}{\eta} = \left(\frac{225}{A} \right)^{1/4} \frac{R_{\ell}}{R_{\lambda}} = 15^{1/4} R_{\lambda}^{1/2}.$$  (3.2.18)
Figure 2.1. Correlated and uncorrelated fluctuations. The fluctuating variable $a$ has the same sign as the variable $b$ for most of the time; this makes $ab > 0$. The variable $c$, on the other hand, is uncorrelated with $a$ and $b$, so that $ac = 0$ and $bc = 0$ (note that $ab \neq 0$, $ac \neq 0$ does not necessarily imply that $bc \neq 0$).
Figure 2.2. Pure shear flow. $U_2 = U_3 = 0$ and all derivatives with respect to $x_1$ and $x_3$ vanish.
Figure 2.3. Molecular motion in a shear flow.
Figure 2.4. Turbulent pure shear flow. The mean velocity is steady: $U_2 = U_3 = 0$ and $U_1 = U_1(x_2)$. The instantaneous streamline pattern sketched refers to a coordinate system that moves with a velocity $U_1(0)$. 

Figs. 18.1a, 1b, 1c, 1d. Turbulent flow in a water channel 6 cm wide, photographed with varying camera speeds. Photographs taken by Nikuradse [39] and published by Tollmien [57].
Figure 2.5. Three-dimensional eddies (vortices with vorticity $\omega$) being stretched by the rate of strain $S$. The fluctuating velocity has strong components in the plane normal to the vorticity vector. Note that the shape of these eddies may differ widely from flow to flow.
Vorticity equation:

\[ \frac{\partial \tilde{\omega}}{\partial t} = -(\tilde{V} \cdot \nabla)\tilde{\omega} - \tilde{\omega}(\nabla \cdot \tilde{V}) + (\nabla \rho \times \nabla \tilde{p})/\rho^2 + (\tilde{\omega} \cdot \nabla)\tilde{V} \]

I II III IV

I: Convection II: Dilatation

III: Baroclinic Torque IV: Vortex Stretching

\[ \tilde{\omega} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \hat{i}_1 + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \hat{i}_2 + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \hat{i}_3 = \omega_1 \hat{i}_1 + \omega_2 \hat{i}_2 + \omega_3 \hat{i}_3 \]

\[ x_1 : \omega_1 \frac{\partial u_1}{\partial x_1} + \omega_2 \frac{\partial u_1}{\partial x_2} + \omega_3 \frac{\partial u_1}{\partial x_3}, \quad x_2 : \omega_1 \frac{\partial u_2}{\partial x_1} + \omega_2 \frac{\partial u_2}{\partial x_2} + \omega_3 \frac{\partial u_2}{\partial x_3} \]

\[ x_3 : \omega_1 \frac{\partial u_3}{\partial x_1} + \omega_2 \frac{\partial u_3}{\partial x_2} + \omega_3 \frac{\partial u_3}{\partial x_3} \]
Figure 2.6. Transport of momentum by turbulent motion.
Figure 2.7. The Lagrangian correlation curve. Some correlation curves have negative tails, many do not.
Figure 2.8. Turbulent flow near a rigid surface with mass transfer. The surface is at rest ($U_1(0) = 0$).
Fig. 8.8  Turbulent shear stress (Reynolds stress) for fully developed turbulent flow in a pipe. (Data from [5].)

\[ \tau = \tau_{\text{lam}} + \tau_{\text{turb}} = \mu \frac{d \bar{u}}{d y} - \rho u' v' \]  

(8.17)
Figure 7-21 Detailed distribution of Reynolds stress in the wall region of a boundary layer. (From Schubauer, 1954.)
Fig. 7-4. Turbulent parallel flow past a plane boundary. (a) Velocity profile; (b) shear stress distribution.
Figure 7-4 Universal wall law plot for turbulent boundary layers on smooth, solid surfaces. (From Clauser, 1956.)
FIGURE 6-9
Replot of the velocity profiles of Fig. 6-8 using inner-law variables $y^+$ and $u^+$. 
Figure 2.9. The velocity scale of flow near a rigid wall with mass transfer (based on data collected by Tennekes, 1965).
Figure 3.1. Stresses on a small volume element in a pure shear flow.
Figure 3.2. The shading pattern used in this book: (a) was selected because it is an isotropic random field, like the small-scale structure of turbulence. The other patterns, (b) and (c), have preferred directions; they are not isotropic.
Figure 3.3. Geometry of wind-tunnel turbulence. The mean flow velocity $U_1$ is independent of $x_1$, but $\alpha^2$ decreases downstream because of viscous dissipation.